

On a three level two-grid finite element method for the 2D-transient Navier-Stokes equations

Saumya Bajpai

TIFR Centre for Applicable Mathematics,
Post Bag No. 6503, GKVK Post Office, Sharada Nagar,
Chikkabommasandra, Bangalore 560065, India,

Amiya K. Pani

Department of Mathematics, Industrial Mathematics Group,
Indian Institute of Technology Bombay, Powai, Mumbai-400076, India

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Abstract

In this paper, an error analysis of a three steps two level Galekin finite element method for the two dimensional transient Navier-Stokes equations is discussed. First of all, the problem is discretized in spatial direction by employing finite element method on a coarse mesh \mathcal{T}_H with mesh size H . Then, in step two, the nonlinear system is linearized around the coarse grid solution, say, u_H , which is similar to Newton's type iteration and the resulting linear system is solved on a finer mesh \mathcal{T}_h with mesh size h . In step three, a correction is obtained through solving a linear problem on the finer mesh and an updated final solution is derived. Optimal error estimates in $L^\infty(\mathbf{L}^2)$ -norm, when $h = \mathcal{O}(H^{2-\delta})$ and in $L^\infty(\mathbf{H}^1)$ -norm, when $h = \mathcal{O}(H^{4-\delta})$ for the velocity and in $L^\infty(L^2)$ -norm, when $h = \mathcal{O}(H^{4-\delta})$ for the pressure are established for arbitrarily small $\delta > 0$. Further, under uniqueness assumption, these estimates are proved to be valid uniformly in time. Then based on backward Euler method, a completely discrete scheme is analyzed and *a priori* error estimates are derived. Finally, the paper is concluded with some numerical experiments.

Keywords: *Two-grid method, 2D-Navier-Stokes system, semidiscrete scheme, backward Euler method, optimal error estimates, order of convergence, uniform-in-time estimates, uniqueness assumption, numerical experiments.*

1 Introduction

Consider the 2D-transient Navier-Stokes system:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t) \quad \mathbf{x} \in \Omega, \quad t > 0, \quad (1.1)$$

and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \quad \mathbf{x} \in \Omega, \quad t > 0, \quad (1.2)$$

with initial and boundary conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \quad \mathbf{u} = 0 \quad \text{on } \partial\Omega, \quad t \geq 0, \quad (1.3)$$

where, Ω is a bounded and convex polygonal domain in \mathbb{R}^2 with boundary $\partial\Omega$ and f is given external force. Here, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ is the velocity vector, $p = p(\mathbf{x}, t)$ denotes the pressure and $\nu > 0$ is the kinematic coefficient of viscosity.

In this article, a three level two-grid finite element Galerkin method for the problem (1.1)-(1.3) is analyzed. The algorithm used here is a suitable modification of the algorithms in [5, 15] and it is composed of the following three steps:

- **Step 1:** solve a nonlinear problem over a coarse mesh with mesh size H which provides an approximate solution, say \mathbf{u}_H .
- **Step 2:** linearize the nonlinear system around the coarse grid solution \mathbf{u}_H and solve the resulting linearized problem over a fine mesh with mesh size h and denote its solution as \mathbf{u}_h^* .
- **Step 3:** correct the solution \mathbf{u}_h^* obtained in **Step 2** over fine mesh which provides an updated final solution \mathbf{u}_h .

As a result of the above mentioned three steps algorithm, the error $\|\mathbf{u} - \mathbf{u}_h\|$ is of the same order as $\|\mathbf{u} - \tilde{\mathbf{u}}_h\|$, where $\tilde{\mathbf{u}}_h$ is the solution of the standard Galerkin system on a fine mesh h with an appropriate scaling between h and H .

The two grid method has been extensively studied for Navier-Stokes equations by Layton [18], Layton and Tobiska [15], Layton and Lenferink [16]-[17], Girault and Lions [8, 9], Dai *et al.* [5], Abboud *et al.* [3]-[4], Frutos *et al.* [7].

In [15], Layton *et al.* have examined a coarse mesh correction in the third step for a steady state Navier-Stokes equations. But, this correction fails to improve the results obtained in **Step 2** and as a result, optimal error estimate in L^2 -norm for the velocity is obtained when $h = \mathcal{O}(H^{3/2})$. Based on stream function formulation, a two-grid finite element method has been studied by Fairag [6]. All the above results have been discussed for the steady state Navier-Stokes equations on a convex polyhedra or on a convex polygon. Subsequently, Girault *et al* [8] in their work on steady state Navier-Stokes equations have analyzed a two level two-grid algorithm and have obtained optimal \mathbf{H}^1 -norm error estimate for the velocity vector with a choice $h = \mathcal{O}(H^2)$, when the problem is defined on a Lipschitz polyhedron or on a convex polyhedron. The analysis is further extended to the transient Navier-Stokes equations in [9], and optimal error estimate in $L^\infty(\mathbf{H}^1)$ -norm is established with a choice $h = \mathcal{O}(H^2)$, when Ω is a Lipschitz polyhedron or a convex polyhedron. In both of these articles, the key approach is to exploit the contribution of the coarse grid solution in $L^3(\Omega)$ -norm.

In the context of nonlinear Galerkin method, two grid method is applied to the 2D-transient Navier-Stokes equations by Ait Ou Amni and Marion in [1]. They have shown that the nonlinear Galerkin solution has the same accuracy as that of the standard Galerkin solution, both for velocity in H^1 -norm and for pressure in L^2 -norm with a choice $h = \mathcal{O}(H^2)$. Further, they have penalized their two-grid algorithm to get rid of the coupling between velocity and pressure with penalization parameter ϵ and have recovered the same accuracy for the penalized two-grid Galerkin solution as that of the standard Galerkin solution with $h = \mathcal{O}(H^2)$ and $h = \mathcal{O}(\epsilon^{1/2})$.

García-Archilla and Titi in [2] have applied Post-Processed method to the semilinear scalar elliptic equations in any dimensions and have derived optimal error bounds in \mathbf{H}^1 -norm for the

post-processed solution with a choice $h = \mathcal{O}(H^{r+1}|\log(H)|^{1/r})$, where the post-processed solution is approximated by the polynomials of degree r with $r \geq 2$.

Recently, Frutos *et al* [7] have applied the two-grid scheme to the incompressible Navier-Stokes equations using mixed-finite elements, the mini-element, the quadratic and the cubic Hood-Taylor elements for spatial discretization and a backward Euler method and a two step backward difference scheme for time discretization and have derived the rate of convergence of the fine mesh in the \mathbf{H}^1 -norm by taking $h = \mathcal{O}(H^2)$, which is an improvement over $h = \mathcal{O}(H^{3/2})$ obtained in [4].

In [20], a fully discrete two-level method consisting of Crank-Nicolson extrapolation method with solution (\mathbf{u}_H, p_H) on a space-time coarse grid J_{H, τ_0} and a backward Euler method with solution (\mathbf{u}_h, p_h) on a space-time fine grid $J_{h, \tau}$ is discussed. They have obtained convergence rate for the two level solution (\mathbf{u}_h, p_h) , which is of same order as that of the one level standard Crank-Nicolson extrapolation solution if $\tau_0^{3/2} + H^{3/2} = \mathcal{O}(\tau)$ for $t \in [0, 1]$ and $\tau_0^2 + H^2 = \mathcal{O}(\tau)$ for $t \in [1, T]$.

An attempt has been made in this article to discuss optimal error estimates in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$ -norms for the velocity and $L^\infty(L^2)$ -norm for the pressure using a three level two-grid finite element method for the 2D-transient Navier-Stokes equations. The major contributions are given in terms of the following two tables. In Table 1, we present the order of convergences for the two-grid algorithm (3.3)-(3.5) stated in Section 3 for the pair of finite element spaces (\mathbf{H}_μ, L_μ) , $\mu = H, h$ satisfying the approximation properties mentioned in (B1)-(B2). Table 2 provides the largest scaling between coarse and fine meshes for which the desired fine mesh accuracy is obtained for both velocity and pressure.

Solution	Velocity in \mathbf{L}^2 -norm	Velocity in \mathbf{H}^1 -norm	Pressure in L^2 -norm
$(\mathbf{u} - \mathbf{u}_H, p - p_H)$	H^2	H	H
$(\mathbf{u} - \mathbf{u}_h^*, p - p_h^*)$	$h^2 + H^{3-\delta}$	$h + H^{3-\delta}$	$h + H^{3-\delta}$
$(\mathbf{u} - \mathbf{u}_h, p - p_h)$	$h^2 + H^{4-2\delta}$	$h + H^{4-\delta}$	$h + H^{4-\delta}$

Table 1: Error estimates obtained from the two-grid algorithm for arbitrarily small $\delta > 0$.

Solution	Velocity in \mathbf{L}^2 -norm	Velocity in \mathbf{H}^1 -norm	Pressure in L^2 -norm
$(\mathbf{u} - \mathbf{u}_H, p - p_H)$	H^2	H	H
$(\mathbf{u} - \mathbf{u}_h^*, p - p_h^*)$	$h \sim H^{(3-\delta)/2}$	$h \sim H^{3-\delta}$	$h \sim H^{3-\delta}$
$(\mathbf{u} - \mathbf{u}_h, p - p_h)$	$h \sim H^{2-\delta}$	$h \sim H^{4-\delta}$	$h \sim H^{4-\delta}$

Table 2: The largest scaling for optimal error estimates.

It is observed from Tables 1 and 2 that the introduction of Step 3 leads to a good improvement in scaling between H and h for both \mathbf{H}^1 -norm for the velocity and L^2 -norm for the pressure, that is, the scaling improves from $h \sim H^{3-\delta}$ to $h \sim H^{4-\delta}$. It also improves the scaling for \mathbf{L}^2 -norm of the velocity from Step 2 to Step 3 from $h \sim H^{(3-\delta)/2}$ to $h \sim H^{2-\delta}$ for arbitrarily small $\delta > 0$.

The main contributions of this paper can be summarized as follows:

- (i) Based on the steady state Oseen projection and Sobolev estimates in Lemma 3.1 involving δ , optimal error estimates for the two-grid Galerkin approximations to the velocity in $L^\infty(\mathbf{H}^1)$ -norm and to the pressure in $L^\infty(L^2)$ -norm with the largest scaling between H and h , $h \sim H^{3-\delta}$ and $h \sim H^{4-\delta}$ for **Step 2** and **Step 3**, respectively are derived. The result obtained in **Step 2** is an improvement over the result obtained by Frutos *et al* [7]. They have obtained using first order mini-elements the largest scaling between H and h , as $h \sim H^2$ for both $L^\infty(\mathbf{H}^1)$ -norm for the velocity and $L^\infty(L^2)$ -norm for the pressure.
- (ii) A use of linearized backward Oseen problem with related estimates yields optimal $L^\infty(\mathbf{L}^2)$ -norm estimates for the velocity in **Step 2** with a choice $h = \mathcal{O}(H^{(3-\delta)/2})$ and **Step 3** with a choice $h = \mathcal{O}(H^{2-\delta})$ for $\delta > 0$ arbitrarily small.
- (iii) Under the assumption of uniqueness condition, *a priori* error estimates are obtained which hold uniformly in time.

The remaining part of the paper consists of the following sections. In Section 2, some preliminaries to be used in the subsequent sections are presented. In Section 3, semidiscrete two-grid finite element approximations are introduced. Optimal error estimates for velocity and pressure are established in Section 4. Section 5 deals with the backward Euler method applied to the semidiscrete two grid system. Finally, in Section 6, the results of some numerical examples which confirm our theoretical results are presented.

2 Preliminaries

We denote \mathbb{R}^2 -valued function spaces using bold face letters, that is, $\mathbf{H}_0^1 = (H_0^1(\Omega))^2$, $\mathbf{L}^2 = (L^2(\Omega))^2$ and $\mathbf{H}^m = (H^m(\Omega))^2$. The standard notations for Lebesgue and Sobolev spaces with their norms are employed in the paper. The space \mathbf{H}_0^1 is equipped with a norm $\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^2 (\partial_j v_i, \partial_j v_i) \right)^{1/2}$. Given a Banach space X endowed with norm $\|\cdot\|_X$, let $L^p(0, T; X)$ be the space of all strongly measurable functions $\phi : [0, T] \rightarrow X$ satisfying $\int_0^T \|\phi(s)\|_X^p ds < \infty$ and for $p = \infty$, $\text{ess sup}_{t \in [0, T]} \|\phi(t)\|_X < \infty$. Also, define

$$\begin{aligned} \mathbf{J} &= \{ \phi \in \mathbf{L}^2 : \nabla \cdot \phi = 0 \text{ in } \Omega, \phi \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly} \} \\ \mathbf{J}_1 &= \{ \phi \in \mathbf{H}_0^1 : \nabla \cdot \phi = 0 \}, \end{aligned}$$

where \mathbf{n} is the unit outward normal to the boundary $\partial\Omega$ and $\phi \cdot \mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [21]. Let H^m/\mathbb{R} be the quotient space with norm $\|\phi\|_{H^m/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|\phi + c\|_m$. For $m = 0$, it is denoted by L^2/\mathbb{R} .

Throughout this paper, we make the following assumptions:

(A1). For $\mathbf{g} \in \mathbf{L}^2$, let $(\mathbf{v}, q) \in \mathbf{J}_1 \times L^2/\mathbb{R}$ be the unique solution to the steady state Stokes problem

$$\begin{aligned} -\Delta \mathbf{v} + \nabla q &= \mathbf{g}, \\ \nabla \cdot \mathbf{v} &= 0 \text{ in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0 \end{aligned}$$

satisfying the regularity result [21]:

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/\mathbb{R}} \leq C \|\mathbf{g}\|.$$

It is easy to show that

$$\|\mathbf{v}\|^2 \leq \lambda_1^{-1} \|\nabla \mathbf{v}\|^2 \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad (2.4)$$

where λ_1 is the minimum eigenvalue of the Laplacian with zero Dirichlet boundary condition.

(A2). There exists a positive constant M_0 such that the initial velocity \mathbf{u}_0 and external force \mathbf{f} satisfy for $t \in (0, T]$ with $0 < T < \infty$

$$\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2, \quad \mathbf{f}, \mathbf{f}_t \in L^\infty(0, T; \mathbf{L}^2) \quad \text{with } \|\mathbf{u}_0\|_2 \leq M_0, \quad \operatorname{ess\,sup}_{0 < t \leq T} \{\|\mathbf{f}(\cdot, t)\|, \|\mathbf{f}_t(\cdot, t)\|\} \leq M_0.$$

Now, for $\mathbf{v}, \mathbf{w}, \phi \in \mathbf{H}_0^1$, define $a(\mathbf{v}, \phi) := (\nabla \mathbf{v}, \nabla \phi)$ and $b(\mathbf{v}, \mathbf{w}, \phi) := \frac{1}{2}(\mathbf{v} \cdot \nabla \mathbf{w}, \phi) - \frac{1}{2}(\mathbf{v} \cdot \nabla \phi, \mathbf{w})$. The weak formulation of (1.1)-(1.3) is to find $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1 \times L^2/\mathbb{R}$, such that $\mathbf{u}(0) = \mathbf{u}_0$ and for $t > 0$

$$\left. \begin{aligned} (\mathbf{u}_t, \phi) + \nu a(\mathbf{u}, \phi) + b(\mathbf{u}, \mathbf{u}, \phi) - (p, \nabla \cdot \phi) &= (\mathbf{f}, \phi) \quad \forall \phi \in \mathbf{H}_0^1, \\ (\nabla \cdot \mathbf{u}, \chi) &= 0 \quad \forall \chi \in L^2. \end{aligned} \right\} \quad (2.5)$$

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that for $\mathbf{u}(0) = \mathbf{u}_0$, $t > 0$,

$$(\mathbf{u}_t, \phi) + \nu a(\mathbf{u}, \phi) + b(\mathbf{u}, \mathbf{u}, \phi) = (\mathbf{f}, \phi) \quad \forall \phi \in \mathbf{J}_1. \quad (2.6)$$

We recall below, the following regularity results.

Lemma 2.1. [10, pp. 285, 302] *Let the assumptions (A1)-(A2) hold true. Then, for any T with $0 < T < \infty$, for any fixed $\alpha > 0$ and for some constant $C = C(M_0)$, the solution of (2.5) satisfies*

$$\begin{aligned} \sup_{0 < t \leq T} \{\|\mathbf{u}(t)\|_2 + \|\mathbf{u}_t(t)\| + \|p(t)\|_{\mathbf{H}^1/\mathbb{R}}\} &\leq C, \\ \sigma^{-1}(t) \int_0^t e^{2\alpha\tau} (\|\mathbf{u}(\tau)\|_2^2 + \|p(\tau)\|_1^2) d\tau &\leq C, \\ \sup_{0 < t \leq T} e^{-2\alpha t} \int_0^t e^{2\alpha\tau} \|\mathbf{u}_t(\tau)\|_1^2 d\tau &\leq C, \quad \sup_{0 < t \leq T} \tau(t) \|\mathbf{u}_t(t)\|_1^2 \leq C, \\ \sup_{0 < t \leq T} e^{-2\alpha t} \int_0^t \sigma(\tau) (\|\mathbf{u}_t(\tau)\|_2^2 + \|\mathbf{u}_{\tau\tau}(\tau)\|^2 + \|p_\tau(\tau)\|_{\mathbf{H}^1/\mathbb{R}}^2) d\tau &\leq C, \end{aligned}$$

where $\tau(t) := \min\{t, 1\}$ and $\sigma(t) := \tau(t)e^{2\alpha t}$.

3 Two-Grid Formulation

Consider two admissible shape regular finite triangulations of $\bar{\Omega}$: a coarse mesh \mathcal{T}_H with mesh size H and a fine mesh \mathcal{T}_h with mesh size h , where $h \ll H$. Let \mathbf{H}_μ and L_μ be the finite dimensional subspaces of \mathbf{H}_0^1 and L^2 , respectively, where $\mu = H, h$. Let us also consider the associated divergence free subspaces \mathbf{J}_μ of \mathbf{H}_μ , where $\mathbf{J}_\mu = \{\phi_\mu \in \mathbf{H}_\mu : (\nabla \cdot \phi_\mu, \chi_\mu) = 0 \quad \forall \chi_\mu \in L_\mu\}$. Note that \mathbf{J}_μ is not a subspace of \mathbf{J}_1 .

Let the spaces \mathbf{H}_μ and L_μ satisfy the following properties:

(B1). (*Approximation property*) For $\mathbf{w} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$, there exist approximations $i_\mu \mathbf{w} \in \mathbf{H}_\mu$ and $j_\mu q \in L_\mu$, such that

$$\|\mathbf{w} - i_\mu \mathbf{w}\| + \mu \|\nabla(\mathbf{w} - i_\mu \mathbf{w})\| \leq C\mu^2 \|\mathbf{w}\|_2, \quad \|q - j_\mu q\|_{L^2/\mathbb{R}} \leq C\mu \|q\|_{H^1/\mathbb{R}}.$$

(B2). (*Uniform inf-sup condition*) There exists a positive constant C , independent of μ , such that

$$\sup_{\phi_\mu \in \mathbf{H}_\mu \setminus \{0\}} \frac{|(q_\mu, \nabla \cdot \phi_\mu)|}{\|\nabla \phi_\mu\|} \geq C \|q_\mu\|_{L_\mu/N_\mu} \quad \forall q_\mu \in L_\mu,$$

where $N_\mu = \{q_\mu \in L_\mu : \forall \phi_\mu \in \mathbf{H}_\mu, (q_\mu, \nabla \cdot \phi_\mu) = 0\}$.

Note that \mathbf{J}_μ is not a subspace of \mathbf{J}_1 . With $P : \mathbf{L}^2 \rightarrow \mathbf{J}$ an orthogonal projection, set the Stokes operator $\tilde{\Delta} = P\Delta$. The L^2 projection $P_\mu : \mathbf{L}^2 \rightarrow \mathbf{J}_\mu$ satisfies the following properties [10]:

$$\left. \begin{aligned} \|\phi - P_\mu \phi\| + \mu \|\nabla P_\mu \phi\| &\leq C\mu \|\nabla \phi\| & \phi \in \mathbf{J}_1, \\ \|\phi - P_\mu \phi\| + \mu \|\nabla(\phi - P_\mu \phi)\| &\leq C\mu^2 \|\tilde{\Delta} \phi\| & \phi \in \mathbf{J}_1 \cap \mathbf{H}^2. \end{aligned} \right\} \quad (3.1)$$

Define the discrete analogue of the Stokes operator as $\tilde{\Delta}_\mu = P_\mu \Delta_\mu$, where Δ_μ is defined by $(\Delta_\mu \mathbf{v}_\mu, \phi_\mu) = -(\nabla \mathbf{v}_\mu, \nabla \phi_\mu)$, for all $\mathbf{v}_\mu, \phi_\mu \in \mathbf{H}_\mu$. Define the 'discrete' Sobolev norms on \mathbf{J}_μ (see [10]) as for $r \in \mathbb{R}$ and for $\mathbf{v}_\mu \in \mathbf{J}_\mu$, $\|\mathbf{v}_\mu\|_r := \|(-\tilde{\Delta}_\mu)^{r/2} \mathbf{v}_\mu\|$.

The operator $b(\cdot, \cdot, \cdot)$ satisfies the antisymmetric property; that is,

$$b(\mathbf{v}_\mu, \mathbf{w}_\mu, \mathbf{w}_\mu) = 0 \quad \forall \mathbf{v}_\mu, \mathbf{w}_\mu \in \mathbf{H}_\mu. \quad (3.2)$$

In the following lemma, we state without proof some estimates of the trilinear term $b(\cdot, \cdot, \cdot)$. For a proof, see [11, pp 360] and [15, pp. 2044].

Lemma 3.1. *The trilinear form $b(\cdot, \cdot, \cdot)$ satisfies the following estimates:*

$$|b(\phi, \xi, \chi)| \leq C \begin{cases} \|\nabla \phi\|^{1/2} \|\tilde{\Delta}_\mu \phi\|^{1/2} \|\nabla \xi\| \|\chi\|, & \text{for all } \phi, \xi, \chi \in \mathbf{H}_\mu, \\ \|\nabla \phi\| \|\nabla \xi\|^{1/2} \|\tilde{\Delta}_\mu \xi\|^{1/2} \|\chi\|, & \text{for all } \phi, \xi, \chi \in \mathbf{H}_\mu, \\ \|\phi\| \|\nabla \xi\| \|\tilde{\Delta} \chi\|, & \text{for all } \phi, \xi \in \mathbf{H}_0^1, \chi \in \mathbf{H}_0^1 \cap \mathbf{H}^2, \\ \|\nabla \phi\| \|\xi\| \|\tilde{\Delta} \chi\|, & \text{for all } \phi, \xi \in \mathbf{H}_0^1, \chi \in \mathbf{H}_0^1 \cap \mathbf{H}^2, \\ \|\nabla \phi\| \|\nabla \xi\| \|\nabla \chi\|, & \text{for all } \phi, \xi, \chi \in \mathbf{H}_0^1, \\ \|\phi\|^{1-\delta} \|\nabla \phi\|^\delta \|\nabla \xi\| \|\nabla \chi\|, & \text{for all } \phi, \xi, \chi \in \mathbf{H}_0^1, \end{cases}$$

where $\delta > 0$ is arbitrarily small.

The three level two-grid semidiscrete algorithm applied to (1.1)-(1.3) is described as follows:

Step 1 (*nonlinear system (1.1) on a coarse grid*): Find $\mathbf{u}_H \in \mathbf{J}_H$ such that for all $\phi_H \in \mathbf{J}_H$ for $\mathbf{u}_H(0) = P_H \mathbf{u}_0$ and $t > 0$

$$(\mathbf{u}_{Ht}, \phi_H) + \nu a(\mathbf{u}_H, \phi_H) + b(\mathbf{u}_H, \mathbf{u}_H, \phi_H) = (\mathbf{f}, \phi_H). \quad (3.3)$$

Step 2 (*Update on a finer mesh with one Newton iteration*) : Seek $\mathbf{u}_h^* \in \mathbf{J}_h$ such that for all $\phi_h \in \mathbf{J}_h$ for $\mathbf{u}_h^*(0) = P_h \mathbf{u}_0$ and $t > 0$

$$(\mathbf{u}_{ht}^*, \phi_h) + \nu a(\mathbf{u}_h^*, \phi_h) + b(\mathbf{u}_h^*, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \phi_h) = (\mathbf{f}, \phi_h) + b(\mathbf{u}_H, \mathbf{u}_H, \phi_h). \quad (3.4)$$

Step 3 (*Correction on a fine mesh*) : Find $\mathbf{u}_h \in \mathbf{J}_h$ such that for all $\phi_h \in \mathbf{J}_h$ for $\mathbf{u}_h(0) = P_h \mathbf{u}_0$ and $t > 0$

$$\begin{aligned} &(\mathbf{u}_{ht}, \phi_h) + \nu a(\mathbf{u}_h, \phi_h) + b(\mathbf{u}_h, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{u}_h, \phi_h) \\ &= (\mathbf{f}, \phi_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \phi_h) + b(\mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}_h^*, \phi_h). \end{aligned} \quad (3.5)$$

The following inequality will be used frequently in our error analysis:

$$\inf_{\phi_h \in \mathbf{J}_h} \sup_{\mathbf{v}_h \in \mathbf{J}_h} \frac{\nu a(\phi_h, \mathbf{v}_h) + b(\mathbf{u}_H, \phi_h, \mathbf{v}_h) + b(\phi_h, \mathbf{u}_H, \mathbf{v}_h)}{\|\nabla \phi_h\| \|\nabla \mathbf{v}_h\|} \geq \gamma > 0. \quad (3.6)$$

For a proof, see [15].

For uniform estimates in time, we shall further assume the following uniqueness condition:

$$\frac{N}{\nu^2} \|\mathbf{f}\|_{L^\infty(0, \infty; \mathbf{L}^2)} < 1 \quad \text{and} \quad N = \sup_{\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{u}, \mathbf{v}, \mathbf{w})}{\|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| \|\nabla \mathbf{w}\|}, \quad (3.7)$$

The main results of this section are stated in the following theorems.

Theorem 3.1. *Let Ω be a convex polygon and let assumptions (A1)-(A2) and (B1)-(B2) hold true. Further, let the discrete initial velocity $\mathbf{u}_{0h} \in \mathbf{J}_h$ with $\mathbf{u}_{0h} = P_h \mathbf{u}_0$. Then, there exists a positive constant C , independent of h , such that for $t \in (0, T]$ with $0 < T < \infty$, the following estimates hold true:*

$$\|(\mathbf{u} - \mathbf{u}_h)(t)\| \leq K(t)(h^2 + H^{4-2\delta}), \quad \|\nabla(\mathbf{u} - \mathbf{u}_h)(t)\| \leq K(t)(h + H^{4-\delta}), \quad (3.8)$$

and

$$\|(p - p_h)(t)\| \leq K(t)(h + H^{4-\delta}), \quad (3.9)$$

where $\delta > 0$ is arbitrarily small and $K(t) = Ce^{Ct}$. Under uniqueness condition (3.7), $K(t) = C$ and the estimates in Theorem 3.1 are valid uniformly in time.

The remaining part of this paper is devoted to the derivation of results, which will lead to the proof of Theorem 3.1.

4 Error Estimates

This section deals with optimal error estimates of the semidiscrete two-grid algorithm. Since \mathbf{J}_h is not a subspace of \mathbf{J}_1 , the weak solution \mathbf{u} satisfies

$$(\mathbf{u}_t, \phi_h) + \nu a(\mathbf{u}, \phi_h) + b(\mathbf{u}, \mathbf{u}, \phi_h) = (\mathbf{f}, \phi_h) + (p, \nabla \cdot \phi_h) \quad \forall \phi_h \in \mathbf{J}_h. \quad (4.10)$$

Define $\mathbf{e}_H := \mathbf{u} - \mathbf{u}_H$, $\mathbf{e}^* := \mathbf{u} - \mathbf{u}_h^*$ and $\mathbf{e}_h := \mathbf{u} - \mathbf{u}_h$. Then, a use of (3.4) and (4.10) yields

$$\begin{aligned} & (\mathbf{e}_t^*, \phi_h) + \nu a(\mathbf{e}^*, \phi_h) + b(\mathbf{e}^*, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{e}^*, \phi_h) \\ & = -b(\mathbf{e}_H, \mathbf{e}_H, \phi_h) + (p, \nabla \cdot \phi_h) \quad \forall \phi_h \in \mathbf{J}_h. \end{aligned} \quad (4.11)$$

Subtract (3.5) from (3.4) and then add the resulting equation to (4.11) to arrive at

$$\begin{aligned} & (\mathbf{e}_{ht}, \phi_h) + \nu a(\mathbf{e}_h, \phi_h) + b(\mathbf{e}_h, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{e}_h, \phi_h) = -b(\mathbf{e}_H, \mathbf{e}^*, \phi_h) \\ & \quad - b(\mathbf{e}^*, \mathbf{e}_H, \phi_h) + b(\mathbf{e}^*, \mathbf{e}^*, \phi_h) + (p, \nabla \cdot \phi_h) \quad \forall \phi_h \in \mathbf{J}_h. \end{aligned} \quad (4.12)$$

For analyzing optimal error estimates of \mathbf{e}_h in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$ -norms, define an auxiliary projection $\tilde{\mathbf{u}}_h(t) \in \mathbf{J}_h$, $0 < t \leq T$, for a given \mathbf{u} , as a solution of the following modified steady state Oseen problem:

$$\nu a(\mathbf{u} - \tilde{\mathbf{u}}_h, \phi_h) + b(\mathbf{u}_H, \mathbf{u} - \tilde{\mathbf{u}}_h, \phi_h) + b(\mathbf{u} - \tilde{\mathbf{u}}_h, \mathbf{u}_H, \phi_h) = (p, \nabla \cdot \phi_h) \quad \text{for all } \phi_h \in \mathbf{J}_h. \quad (4.13)$$

Now split \mathbf{e}_h as

$$\mathbf{e}_h =: (\mathbf{u} - \tilde{\mathbf{u}}_h) + (\tilde{\mathbf{u}}_h - \mathbf{u}_h) := \boldsymbol{\zeta} + \Theta, \quad (4.14)$$

where $\boldsymbol{\zeta} := \mathbf{u} - \tilde{\mathbf{u}}_h$ and $\Theta := \tilde{\mathbf{u}}_h - \mathbf{u}_h$.

A use of (4.12)-(4.14) leads to

$$\begin{aligned} (\Theta_t, \phi_h) + \nu a(\Theta, \phi_h) + b(\Theta, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \Theta, \phi_h) &= -(\boldsymbol{\zeta}_t, \phi_h) \\ &\quad - b(\mathbf{e}_H, \mathbf{e}^*, \phi_h) - b(\mathbf{e}^*, \mathbf{e}_H, \phi_h) + b(\mathbf{e}^*, \mathbf{e}^*, \phi_h). \end{aligned} \quad (4.15)$$

To seek estimates for Θ , we need estimates for $\boldsymbol{\zeta}$, \mathbf{e}_H and \mathbf{e}^* , which appear on the right hand side of (4.15).

Lemma 4.1. [11, see page 362, proposition 3.2] *Let $\mathbf{u}_H(t)$ be the solution of (3.3) in $[0, T)$, $0 < T < \infty$ satisfying $\mathbf{u}_{0H} = P_H \mathbf{u}_0$, and let the assumptions (A1)-(A2) hold true. Then, there exists a positive constant $C = C(\gamma, \nu, \alpha, \lambda_1, M_0)$ such that the following hold true for all $t > 0$*

$$\|\mathbf{u}_H(t)\|_2 + \|\mathbf{u}_{Ht}(t)\| + e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|\tilde{\Delta} \mathbf{u}_H(s)\|^2 + \|\nabla \mathbf{u}_{Ht}(s)\|^2) ds \leq C, \quad \sup_{0 < t < T} \tau(t) \|\mathbf{u}_{Ht}(t)\|_1^2 \leq C.$$

Lemma 4.2. [10, estimates for \mathbf{e}_H] *Let the assumptions (A1)-(A2) and (B1)-(B2) hold true. With initial velocity $\mathbf{u}_{0H} = P_H \mathbf{u}_0$, let the discrete solution pair $(\mathbf{u}_H(t), p_H(t))$ satisfies (3.3). Then, there exists a positive constant C , independent of H , such that for $0 < t \leq T$*

$$\|\mathbf{e}_H(t)\| \leq K(t) H^2 \quad \|(p - p_H)(t)\| \leq K(t) H$$

and

$$\sigma^{-1}(t) \int_0^t e^{2\alpha \tau} \|\mathbf{e}_H(\tau)\|^2 d\tau \leq K(t) H^4, \quad e^{-2\alpha t} \int_0^t \sigma(\tau) \|\nabla \mathbf{e}_{H\tau}(\tau)\|^2 d\tau \leq K(t) H^2,$$

where $K(t) = C e^{Ct}$. If in addition, uniqueness condition (3.7) holds true, then $K(t) = C$ and the results are valid uniformly in time.

In order to derive estimates for \mathbf{e}^* , split it as

$$\mathbf{e}^* = (\mathbf{u} - \tilde{\mathbf{u}}_h) + (\tilde{\mathbf{u}}_h - \mathbf{u}_h^*) := \boldsymbol{\zeta} + \boldsymbol{\rho}. \quad (4.16)$$

The following lemma provides estimates for $\tilde{\mathbf{u}}_h$. Since a suitable modification of the proofs in [15] will provide a proof, we state below the results without the proofs.

Lemma 4.3. *Let the approximation property (B1) be satisfied. Further, let \mathbf{u} and $\tilde{\mathbf{u}}_h$ be the solution of (2.5) and (4.13), respectively. Then, the following estimates hold true:*

$$\begin{aligned} \|\nabla \boldsymbol{\zeta}(t)\| &\leq Ch \mathcal{K}(t), \quad \|\nabla \boldsymbol{\zeta}_t(t)\| \leq Ch (\mathcal{K}_t(t) + \mathcal{K}(t) \|\nabla \mathbf{u}_{Ht}\|), \\ \|\boldsymbol{\zeta}(t)\| &\leq Ch \mathcal{K}(t)(h + \|\mathbf{e}_H\|), \quad \|\boldsymbol{\zeta}_t(t)\| \leq Ch(h + \|\mathbf{e}_H\|)(\mathcal{K}(t) \|\nabla \mathbf{u}_{Ht}\| + \mathcal{K}_t(t)), \end{aligned}$$

where $\mathcal{K}(t) := \|\tilde{\Delta} \mathbf{u}(t)\| + \|\nabla p(t)\|$ and $\mathcal{K}_t(t) := \|\tilde{\Delta} \mathbf{u}_t(t)\| + \|\nabla p_t(t)\|$.

We note that $\mathcal{K}(t)$ and $\mathcal{K}_t(t)$ satisfy for some positive constant K

$$\sup_{0 < t \leq T} e^{-2\alpha t} \int_0^t e^{2\alpha s} (\mathcal{K}^2(s) + \tau(s) \mathcal{K}_t^2(s)) ds \leq K. \quad (4.17)$$

Below in Subsections 4.1 and 4.2, we focus on the semidiscrete error estimates related to **Step 2** and **Step 3**.

4.1 Error estimates for Step 2

In this subsection, the semidiscrete error estimates corresponding to **Step 2** are derived.

Lemma 4.4. *With $0 \leq \alpha \leq \gamma\lambda_1$ and $\beta = \gamma - \alpha\lambda_1^{-1} > 0$, let the hypothesis of Lemma 4.2 hold true. Then, the following estimate holds:*

$$\beta \sigma^{-1}(t) \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau \leq K(t)(h^2 + H^{6-2\delta}).$$

Here and elsewhere in this paper, $K(t)$ denotes Ce^{Ct} and under uniqueness assumption (3.7), $K(t)$, reduces to a positive constant K .

Proof. Choose $\phi_h = P_h \mathbf{e}^* = (P_h \mathbf{u} - \mathbf{u}) + \mathbf{e}^*$ in (4.11). Then, a use of (3.6) yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{e}^*\|^2 + \gamma \|\nabla \mathbf{e}^*\|^2 &\leq (\mathbf{e}_t^*, \mathbf{u} - P_h \mathbf{u}) + \nu a(\mathbf{e}^*, \mathbf{u} - P_h \mathbf{u}) + b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{u} - P_h \mathbf{u}) \\ &\quad + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{u} - P_h \mathbf{u}) - b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}^*) + (p, \nabla \cdot P_h \mathbf{e}^*). \end{aligned} \quad (4.18)$$

Using the definition of P_h , the first term on the right hand side of (4.18) can be treated as

$$\begin{aligned} (\mathbf{e}_t^*, \mathbf{u} - P_h \mathbf{u}) &= (\mathbf{u}_t - P_h \mathbf{u}_t + P_h \mathbf{u}_t - \mathbf{u}_h^*, \mathbf{u} - P_h \mathbf{u}) \\ &= (\mathbf{u}_t - P_h \mathbf{u}_t, \mathbf{u} - P_h \mathbf{u}) = \frac{1}{2} \frac{d}{dt} \|\mathbf{u} - P_h \mathbf{u}\|^2. \end{aligned} \quad (4.19)$$

An application of the Cauchy-Schwarz's inequality with (3.1) leads to

$$\nu |a(\mathbf{e}^*, \mathbf{u} - P_h \mathbf{u})| \leq C(\nu) \|\nabla \mathbf{e}^*\| \|\nabla(\mathbf{u} - P_h \mathbf{u})\| \leq C(\nu) h \|\tilde{\Delta} \mathbf{u}\| \|\nabla \mathbf{e}^*\|. \quad (4.20)$$

Apply (3.1), Lemma 3.1 and the boundedness of $\|\nabla \mathbf{u}_H\|$ to arrive at

$$\begin{aligned} |b(\mathbf{u}_H, \mathbf{e}^*, \mathbf{u} - P_h \mathbf{u}) + b(\mathbf{e}^*, \mathbf{u}_H, \mathbf{u} - P_h \mathbf{u})| &\leq C \|\nabla \mathbf{u}_H\| \|\nabla \mathbf{e}^*\| \|\nabla(\mathbf{u} - P_h \mathbf{u})\| \\ &\leq Ch \|\tilde{\Delta} \mathbf{u}\| \|\nabla \mathbf{u}_H\| \|\nabla \mathbf{e}^*\| \leq Ch \|\tilde{\Delta} \mathbf{u}\| \|\nabla \mathbf{e}^*\|. \end{aligned} \quad (4.21)$$

The discrete incompressibility condition shows that

$$|(p - j_h p, \nabla \cdot P_h \mathbf{e}^*)| \leq \|p - j_h p\| \|\nabla \mathbf{e}^*\| \leq Ch \|\nabla p\| \|\nabla \mathbf{e}^*\|. \quad (4.22)$$

A use of (3.1) with Lemma 3.1 yields

$$|b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}^*)| \leq C \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^{1+\delta} \|\nabla \mathbf{e}^*\|. \quad (4.23)$$

Substitute (4.19)-(4.23) in (4.18) along with the Young's inequality, (2.4) and multiply the resulting inequality by $e^{2\alpha t}$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} e^{2\alpha t} \|\mathbf{e}^*\|^2 + \frac{1}{2} \left(\gamma - \frac{\alpha}{\lambda_1} \right) e^{2\alpha t} \|\nabla \mathbf{e}^*\|^2 &\leq \frac{1}{2} \frac{d}{dt} e^{2\alpha t} \|\mathbf{u} - P_h \mathbf{u}\|^2 \\ &\quad + C e^{2\alpha t} \left(h^2 \mathcal{K}^2 + \|\mathbf{e}_H\|^{2(1-\delta)} \|\nabla \mathbf{e}_H\|^{2(1+\delta)} \right) - \alpha e^{2\alpha t} \|\mathbf{u} - P_h \mathbf{u}\|^2. \end{aligned} \quad (4.24)$$

The last term in the right hand side of (4.24) is negative. We drop this term. Integrate (4.24) with respect to time, use $\|\mathbf{e}^*\| \leq \|\mathbf{u} - P_h \mathbf{u}\|$, $\|\mathbf{e}^*(0)\| = \|\mathbf{u}_0 - P_h \mathbf{u}_0\|$ and Lemma 2.1 with $\beta = \gamma - \alpha\lambda_1^{-1} > 0$ to arrive at

$$\begin{aligned} \beta \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau &\leq C \int_0^t e^{2\alpha\tau} \left(h^2 \mathcal{K}^2(\tau) + \|\mathbf{e}_H(\tau)\|^{2(1-\delta)} \|\nabla \mathbf{e}_H(\tau)\|^{2(1+\delta)} \right) d\tau \\ &\leq C \left(h^2 \sigma + \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{-2\delta} \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2(1+\delta)} \int_0^t e^{2\alpha\tau} \|\mathbf{e}_H(\tau)\|^2 d\tau \right). \end{aligned} \quad (4.25)$$

An application of Lemma 4.2 in (4.25) completes the proof. \square

Next, we prove $L^\infty(\mathbf{H}^1)$ -norm estimate of \mathbf{e}^* .

Lemma 4.5. *Under the hypotheses of Lemma 4.4, the following estimate holds true:*

$$\sigma^{-1}(t) \int_0^t \sigma(\tau) \|\mathbf{e}_\tau^*\|^2 d\tau + \|\nabla \mathbf{e}^*\|^2 \leq K(t) (h^2 + H^{6-2\delta}).$$

Proof. Substitute $\phi_h = \sigma P_h \mathbf{e}_t^* = \sigma(P_h \mathbf{u}_t - \mathbf{u}_t) + \sigma \mathbf{e}_t^*$ in (4.11) to obtain

$$\begin{aligned} \sigma \|\mathbf{e}_t^*\|^2 + \frac{\nu}{2} \frac{d}{dt} (\sigma \|\nabla \mathbf{e}^*\|^2) &= \sigma(\mathbf{e}_t^*, \mathbf{u}_t - P_h \mathbf{u}_t) + \sigma \nu a(\mathbf{e}^*, \mathbf{u}_t - P_h \mathbf{u}_t) + \frac{\nu}{2} \sigma_t \|\nabla \mathbf{e}^*\|^2 \\ &\quad - \sigma b(\mathbf{u}_H, \mathbf{e}^*, P_h \mathbf{e}_t^*) - \sigma b(\mathbf{e}^*, \mathbf{u}_H, P_h \mathbf{e}_t^*) + \sigma b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}_t^*) + \sigma(p, \nabla \cdot P_h \mathbf{e}_t^*). \end{aligned} \quad (4.26)$$

Now, rewrite

$$\begin{aligned} \sigma b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}_t^*) &= \frac{d}{dt} (\sigma b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}^*)) - \sigma_t b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}^*) - \sigma b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h \mathbf{e}^*) \\ &\quad - \sigma b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h \mathbf{e}^*), \end{aligned} \quad (4.27)$$

and

$$\begin{aligned} \sigma(p, \nabla \cdot P_h \mathbf{e}_t^*) &= \frac{d}{dt} (\sigma(p - j_h p, \nabla \cdot P_h \mathbf{e}^*)) - \sigma_t(p - j_h p, \nabla \cdot P_h \mathbf{e}^*) \\ &\quad - \sigma(p_t - j_h p_t, \nabla \cdot P_h \mathbf{e}^*). \end{aligned} \quad (4.28)$$

An application of Lemma 3.1 with (3.1) leads to

$$\begin{aligned} &|\sigma_t b(\mathbf{e}_H, \mathbf{e}_H, P_h \mathbf{e}^*) + \sigma b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h \mathbf{e}^*) + \sigma b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h \mathbf{e}^*)| \\ &\leq C (\sigma_t \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^{1+\delta} + \sigma \|\nabla \mathbf{e}_{Ht}\| \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^\delta) \|\nabla \mathbf{e}^*\|. \end{aligned} \quad (4.29)$$

Apply (4.27)-(4.29) along with the Cauchy-Schwarz's inequality, the Young's inequality, (3.1) and Lemma 3.1 in (4.26). Then integrate the resulting equation with respect to time from 0 to t to arrive at

$$\begin{aligned} &\int_0^t \sigma(\tau) \|\mathbf{e}_\tau^*\|^2 d\tau + \nu \sigma \|\nabla \mathbf{e}^*\|^2 \leq C(\nu) \left(h^2 \int_0^t (\sigma(\tau) \|\nabla \mathbf{u}_t(\tau)\|^2 + \sigma_\tau(\tau) \|\nabla p(\tau)\|^2) d\tau \right. \\ &\quad + h^2 \int_0^t \frac{\sigma^2(\tau)}{\sigma_\tau(\tau)} \mathcal{K}_\tau^2(\tau) d\tau + \int_0^t \left(e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 + \sigma(\tau) \|\nabla \mathbf{u}_H(\tau)\| \|\tilde{\Delta} \mathbf{u}_H(\tau)\| \|\nabla \mathbf{e}^*(\tau)\|^2 \right) d\tau \\ &\quad + \sigma(\|\mathbf{e}_H\|^{2(1-\delta)} \|\nabla \mathbf{e}_H\|^{2(1+\delta)} + h^2 \|\nabla p\|^2) + \int_0^t \sigma_\tau(\tau) \|\mathbf{e}_H(\tau)\|^{2(1-\delta)} \|\nabla \mathbf{e}_H(\tau)\|^{2(1+\delta)} d\tau \\ &\quad \left. + \int_0^t \frac{\sigma^2(\tau)}{\sigma_\tau(\tau)} \|\nabla \mathbf{e}_{H\tau}(\tau)\|^2 \|\mathbf{e}_H(\tau)\|^{2(1-\delta)} \|\nabla \mathbf{e}_H(\tau)\|^{2\delta} d\tau \right). \end{aligned}$$

A use of $\frac{\sigma(\tau)}{\sigma_\tau(\tau)} \leq C \min\{1, \tau\}$ with $\sigma_\tau(\tau) \leq C e^{2\alpha\tau}$ and $\min\{1, \tau\} \leq \min\{1, t\}$ leads to

$$\begin{aligned} &\int_0^t \sigma(\tau) \|\mathbf{e}_\tau^*\|^2 d\tau + \nu \sigma \|\nabla \mathbf{e}^*\|^2 \leq C(\nu) \left(h^2 \int_0^t (\sigma(\tau) \|\nabla \mathbf{u}_t(\tau)\|^2 + \sigma_\tau(\tau) \|\nabla p(\tau)\|^2) d\tau \right. \\ &\quad + h^2 \min\{1, t\} \int_0^t \sigma(\tau) \mathcal{K}_\tau^2(\tau) d\tau + \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau \\ &\quad + \int_0^t \sigma(\tau) \|\nabla \mathbf{u}_H(\tau)\| \|\tilde{\Delta} \mathbf{u}_H(\tau)\| \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau + \sigma(\|\mathbf{e}_H\|^{2(1-\delta)} \|\nabla \mathbf{e}_H\|^{2(1+\delta)} + h^2 \|\nabla p\|^2) \\ &\quad + \int_0^t \sigma_\tau(\tau) \|\mathbf{e}_H(\tau)\|^{2(1-\delta)} \|\nabla \mathbf{e}_H(\tau)\|^{2(1+\delta)} d\tau \\ &\quad \left. + \min\{1, t\} \int_0^t \sigma(\tau) \|\nabla \mathbf{e}_{H\tau}(\tau)\|^2 \|\mathbf{e}_H(\tau)\|^{2(1-\delta)} \|\nabla \mathbf{e}_H(\tau)\|^{2\delta} d\tau \right). \end{aligned}$$

From Lemmas 2.1, 4.2, 4.4 and boundedness of $\|\nabla \mathbf{u}_H\|$, it follows that

$$\begin{aligned} \int_0^t \sigma(\tau) \|\mathbf{e}_\tau^*(\tau)\|^2 d\tau + \nu \sigma \|\nabla \mathbf{e}^*\|^2 &\leq C \left(h^2 \sigma + \sigma \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2(1-\delta)} \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2(1+\delta)} \right. \\ &\quad + \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau + \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{-2\delta} \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2(1+\delta)} \int_0^t e^{2\alpha\tau} \|\mathbf{e}_H(\tau)\|^2 d\tau \\ &\quad \left. + \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2-2\delta} \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2\delta} \int_0^t \frac{\sigma^2(\tau)}{\sigma_\tau(\tau)} \|\nabla \mathbf{e}_{H\tau}(\tau)\|^2 d\tau \right). \end{aligned} \quad (4.30)$$

An application of Lemmas 4.2, 4.4 completes the proof. \square

Lemma 4.6. *Under the hypotheses of Lemma 4.2, the following estimate holds true:*

$$\sigma^{-1}(t) \int_0^t e^{2\alpha\tau} \|\mathbf{e}^*(\tau)\|^2 d\tau \leq K(t) (h^4 + h^2 H^4 + H^6).$$

Proof. Consider the linearized backward problem [12]. For a given $\mathbf{e}^* \in L^2(\mathbf{L}^2)$, let $(\phi(t), \psi(t)) \in \mathbf{J}_1 \times L^2(\Omega)/\mathbb{R}$ be a weak solution of

$$(\mathbf{v}, \phi_t) - \nu a(\mathbf{v}, \phi) - b(\mathbf{u}, \mathbf{v}, \phi) - b(\mathbf{v}, \mathbf{u}, \phi) + (\psi, \nabla \cdot \mathbf{v}) = (e^{2\alpha t} \mathbf{e}^*, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega). \quad (4.31)$$

with $\phi(T) = 0$, satisfying

$$\int_0^T e^{-2\alpha t} (\|\phi\|_2^2 + \|\psi\|_1^2 + \|\phi_t\|^2) dt \leq C \int_0^T e^{2\alpha t} \|\mathbf{e}^*\|^2 dt. \quad (4.32)$$

Rewrite (4.31) as

$$(\mathbf{v}, \phi_t) - \nu a(\mathbf{v}, \phi) - b(\mathbf{u}_H, \mathbf{v}, \phi) - b(\mathbf{v}, \mathbf{u}_H, \phi) - b(\mathbf{e}_H, \mathbf{v}, \phi) - b(\mathbf{v}, \mathbf{e}_H, \phi) + (\psi, \nabla \cdot \mathbf{v}) = (e^{\alpha t} \mathbf{e}^*, \mathbf{v}).$$

Substitute $\mathbf{v} = \mathbf{e}^*$, use (4.11) with $\phi_h = P_h \phi$ and the discrete incompressibility condition to obtain

$$\begin{aligned} e^{2\alpha t} \|\mathbf{e}^*\|^2 &= \frac{d}{dt} (\mathbf{e}^*, \phi) - (\mathbf{e}_t^*, \phi - P_h \phi) - \nu a(\mathbf{e}^*, \phi - P_h \phi) - b(\mathbf{u}_H, \mathbf{e}^*, \phi - P_h \phi) \\ &\quad - b(\mathbf{e}^*, \mathbf{u}_H, \phi - P_h \phi) - b(\mathbf{e}_H, \mathbf{e}^*, \phi) - b(\mathbf{e}^*, \mathbf{e}_H, \phi) - b(\mathbf{e}_H, \mathbf{e}_H, P_h \phi) \\ &\quad - (p - j_h p, \nabla \cdot (P_h \phi - \phi)) + (\psi - j_h \psi, \nabla \cdot \mathbf{e}^*). \end{aligned}$$

Using the fact

$$\begin{aligned} (\mathbf{e}_t^*, \phi - P_h \phi) &= \frac{d}{dt} (\mathbf{e}^*, \phi - P_h \phi) - (\mathbf{e}^*, \phi_t - P_h \phi_t) \\ &= \frac{d}{dt} (\mathbf{e}^*, \phi - P_h \phi) - (\mathbf{u} - P_h \mathbf{u}, \phi_t), \end{aligned}$$

we now arrive at

$$\begin{aligned} e^{2\alpha t} \|\mathbf{e}^*\|^2 &= \frac{d}{dt} (\mathbf{e}^*, P_h \phi) - (\mathbf{u} - P_h \mathbf{u}, \phi_t) - \nu a(\mathbf{e}^*, \phi - P_h \phi) - b(\mathbf{u}_H, \mathbf{e}^*, \phi - P_h \phi) \\ &\quad - b(\mathbf{e}^*, \mathbf{u}_H, \phi - P_h \phi) - b(\mathbf{e}_H, \mathbf{e}^*, \phi) - b(\mathbf{e}^*, \mathbf{e}_H, \phi) - b(\mathbf{e}_H, \mathbf{e}_H, P_h \phi - \phi) \\ &\quad - b(\mathbf{e}_H, \mathbf{e}_H, \phi) - (p - j_h p, \nabla \cdot (P_h \phi - \phi)) + (\psi - j_h \psi, \nabla \cdot \mathbf{e}^*). \end{aligned}$$

Integrate with respect to time from 0 to T and use $\phi(T) = 0$ to find that

$$\begin{aligned}
\int_0^T e^{2\alpha\tau} \|\mathbf{e}^*(\tau)\|^2 d\tau &= -(\mathbf{e}^*(0), P_h \phi(0)) - \int_0^T (\mathbf{u} - P_h \mathbf{u}, \phi_\tau) d\tau \\
&\quad - \nu \int_0^T a(\mathbf{e}^*, \phi - P_h \phi) d\tau - \int_0^T (b(\mathbf{u}_H, \mathbf{e}^*, \phi - P_h \phi) + b(\mathbf{e}^*, \mathbf{u}_H, \phi - P_h \phi)) d\tau \\
&\quad - \int_0^T ((b(\mathbf{e}_H, \mathbf{e}^*, \phi) + b(\mathbf{e}^*, \mathbf{e}_H, \phi) + b(\mathbf{e}_H, \mathbf{e}_H, P_h \phi - \phi) + b(\mathbf{e}_H, \mathbf{e}_H, \phi)) d\tau \\
&\quad + \int_0^T (-(p - j_h p, \nabla \cdot (P_h \phi - \phi)) + (\psi - j_h \psi, \nabla \cdot \mathbf{e}^*)) d\tau. \tag{4.33}
\end{aligned}$$

The first term in the right hand side of (4.33) vanishes due to the orthogonality property of P_h . An application of (3.1) with Cauchy-Schwarz's inequality and Young's inequality yields

$$\begin{aligned}
\int_0^T |(\mathbf{u} - P_h \mathbf{u}, \phi_\tau) + a(\mathbf{e}^*, \phi - P_h \phi)| d\tau &\leq C(\epsilon) h^2 \int_0^T e^{2\alpha\tau} (h^2 \|\tilde{\Delta} \mathbf{u}\|^2 + \|\nabla \mathbf{e}^*\|^2) d\tau \\
&\quad + \epsilon \int_0^T e^{-2\alpha\tau} \|\phi_\tau\|^2 d\tau \tag{4.34}
\end{aligned}$$

A use of (3.1) with Lemma 3.1 and boundedness of $\|\nabla \mathbf{u}_H\|$ shows

$$\begin{aligned}
\int_0^T |b(\mathbf{u}_H, \mathbf{e}^*, \phi - P_h \phi) + b(\mathbf{e}^*, \mathbf{u}_H, \phi - P_h \phi)| + |b(\mathbf{e}_H, \mathbf{e}^*, \phi) + b(\mathbf{e}^*, \mathbf{e}_H, \phi)| d\tau \\
\leq C(\epsilon) h^2 \int_0^T e^{2\alpha\tau} (h^2 + \|\mathbf{e}_H\|^2) \|\nabla \mathbf{e}^*\|^2 d\tau + \epsilon \int_0^T e^{-2\alpha\tau} \|\phi\|_2^2 d\tau. \tag{4.35}
\end{aligned}$$

Apply (3.1) and **(B2)** to obtain

$$\begin{aligned}
\int_0^T |(p - j_h p, \nabla \cdot (P_h \phi - \phi))| + |(\psi - j_h \psi, \nabla \cdot \mathbf{e}^*)| d\tau &\leq C \int_0^T (h^2 \|\nabla p\| \|\phi\|_2 + h \|\nabla \mathbf{e}^*\| \|\psi\|_1) d\tau \\
&\leq C(\epsilon) \int_0^T e^{2\alpha\tau} (h^4 \|\nabla p\|^2 + h^2 \|\nabla \mathbf{e}^*\|^2) d\tau + \epsilon \int_0^T e^{-2\alpha\tau} \|\phi\|_2^2 d\tau. \tag{4.36}
\end{aligned}$$

A use of (3.1) with Lemma 3.1 leads to

$$\begin{aligned}
\int_0^T (|b(\mathbf{e}_H, \mathbf{e}_H, P_h \phi - \phi)| + |b(\mathbf{e}_H, \mathbf{e}_H, \phi)|) d\tau &\leq C(\epsilon) (h^2 \int_0^T e^{2\alpha\tau} \|\mathbf{e}_H\|^{2(1-\delta)} \|\nabla \mathbf{e}_H\|^{2(1+\delta)} d\tau \\
&\quad + \int_0^T e^{2\alpha\tau} \|\mathbf{e}_H\|^2 \|\nabla \mathbf{e}_H\|^2 d\tau) + \epsilon \int_0^T e^{-2\alpha\tau} \|\phi\|_2^2 d\tau. \tag{4.37}
\end{aligned}$$

Substitute (4.34)-(4.37), regularity results (4.32) and Lemma 4.4 in (4.33) with $\epsilon = 1/4$ to obtain

$$\begin{aligned}
\int_0^T e^{2\alpha\tau} \|\mathbf{e}^*(\tau)\|^2 d\tau &\leq C \left(h^4 \int_0^T e^{2\alpha\tau} \mathcal{K}^2(\tau) d\tau + \int_0^T e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 (h^2 + \|\mathbf{e}_H(\tau)\|^2) d\tau \right. \\
&\quad \left. h^2 \int_0^T e^{2\alpha\tau} \|\mathbf{e}_H(\tau)\|^{2(1-\delta)} \|\nabla \mathbf{e}_H(\tau)\|^{2(1+\delta)} d\tau + \int_0^T e^{2\alpha\tau} \|\mathbf{e}_H(\tau)\|^2 \|\nabla \mathbf{e}_H(\tau)\|^2 d\tau \right) \\
&\leq C \left(h^4 \int_0^T e^{2\alpha\tau} \mathcal{K}^2(\tau) d\tau + \left(h^2 + \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^2 \right) \int_0^T e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau \right. \\
&\quad \left. + \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^2 \int_0^T e^{2\alpha\tau} \|\mathbf{e}_H(\tau)\|^2 d\tau \right). \tag{4.38}
\end{aligned}$$

A use of Lemmas 2.1, 4.2 and 4.4 in (4.38) concludes the proof. \square

The following theorem provides estimates for \mathbf{e}^* .

Theorem 4.1. *Let the assumptions of Theorem 3.1 be satisfied. Further, let the discrete initial initial velocity $\mathbf{u}_{0h}^* \in \mathbf{J}_h$ with $\mathbf{u}_{0h}^* = P_h \mathbf{u}_0$, where $\mathbf{u}_0 \in \mathbf{J}_1$. Then, there exists a positive constant C , independent of h , such that for $t \in (0, T]$ with $0 < T < \infty$, the following estimates hold true:*

$$\|(\mathbf{u} - \mathbf{u}_h^*)(t)\| \leq K(t)(h^2 + H^{3-\delta}), \quad \|\nabla(\mathbf{u} - \mathbf{u}_h^*)(t)\| \leq K(t)(h + H^{3-\delta}),$$

where $\delta > 0$ is arbitrarily small and $K(t) = Ce^{Ct}$. If, in addition, uniqueness condition (3.7) holds true, then $K(t) = K$, that is, estimates are bounded uniformly with respect to time.

Proof. Since $\mathbf{e}^* = \boldsymbol{\zeta} + \boldsymbol{\rho}$ and estimates of $\boldsymbol{\zeta}$ are known from Lemma 4.3, it is enough to derive estimates of $\boldsymbol{\rho}$. A use of (4.11) with (4.13) and (4.16) leads to

$$\begin{aligned} (\boldsymbol{\rho}_t, \boldsymbol{\phi}_h) + \nu a(\boldsymbol{\rho}, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \boldsymbol{\rho}, \boldsymbol{\phi}_h) + b(\boldsymbol{\rho}, \mathbf{u}_H, \boldsymbol{\phi}_h) = & -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) \\ & + b(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\phi}_h) \quad \text{for all } \boldsymbol{\phi}_h \in \mathbf{J}_h. \end{aligned} \quad (4.39)$$

Multiplying (4.39) by $\sigma(t)$, substitute $\boldsymbol{\phi}_h = \boldsymbol{\rho}$ and use (3.6) to arrive at

$$\frac{1}{2} \frac{d}{dt} (\sigma \|\boldsymbol{\rho}\|^2) + \gamma \sigma \|\nabla \boldsymbol{\rho}\|^2 \leq -\sigma (\boldsymbol{\zeta}_t, \boldsymbol{\rho}) + \sigma b(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\rho}) + \frac{1}{2} \sigma_t \|\boldsymbol{\rho}\|^2,$$

where $\sigma(t) = \min\{t, 1\} e^{2\alpha t}$.

Integrate with respect to time from 0 to t and obtain

$$\begin{aligned} \sigma(t) \|\boldsymbol{\rho}(t)\|^2 + \gamma \int_0^t \sigma(\tau) \|\nabla \boldsymbol{\rho}(\tau)\|^2 d\tau \leq & - \int_0^t \sigma(\tau) (\boldsymbol{\zeta}_\tau(\tau), \boldsymbol{\rho}) d\tau \\ & + \int_0^t \sigma(\tau) b(\mathbf{e}_H(\tau), \mathbf{e}_H(\tau), \boldsymbol{\rho}) d\tau + \frac{1}{2} \int_0^t \sigma_\tau(\tau) \|\boldsymbol{\rho}(\tau)\|^2 d\tau. \end{aligned} \quad (4.40)$$

Using the Cauchy-Schwarz inequality and Lemma 4.3, the first term on the right hand side of (4.40) can be bounded as

$$\begin{aligned} \int_0^t \sigma(\tau) (\boldsymbol{\zeta}_\tau(\tau), \boldsymbol{\rho}) d\tau & \leq \int_0^t \sigma(\tau) \|\boldsymbol{\zeta}_\tau(\tau)\| \|\boldsymbol{\rho}\| d\tau \\ & \leq \int_0^t \sigma(\tau) h(h + \|\mathbf{e}_H(\tau)\|) (\mathcal{K}(\tau) \|\nabla \mathbf{u}_{H\tau}(\tau)\| + \mathcal{K}_\tau(\tau)) \|\boldsymbol{\rho}\| d\tau. \end{aligned} \quad (4.41)$$

A use of The Young's inequality with estimates of $\|\nabla \mathbf{u}_{Ht}\|$, Lemmas 2.1 and 4.2 in (4.41) leads to

$$\begin{aligned} \int_0^t \sigma(\tau) (\boldsymbol{\zeta}_\tau(\tau), \boldsymbol{\rho}) d\tau & \leq C(\lambda_1, \epsilon) h^2 (h^2 + \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^2) \left(\sup_{0 < t < T} (\tau(t) \|\nabla \mathbf{u}_{Ht}\|^2) \int_0^t e^{2\alpha\tau} \mathcal{K}^2(\tau) d\tau \right. \\ & \quad \left. + \int_0^t \frac{\sigma^2(\tau)}{\sigma_\tau(\tau)} \mathcal{K}_\tau^2(\tau) d\tau \right) + \int_0^t \sigma_\tau(\tau) \|\boldsymbol{\rho}(\tau)\|^2 d\tau + \epsilon \int_0^t \sigma(\tau) \|\nabla \boldsymbol{\rho}(\tau)\|^2 d\tau. \end{aligned} \quad (4.42)$$

For the second term in the right hand side of (4.41), split $\boldsymbol{\rho} = \mathbf{e}^* - \boldsymbol{\zeta}$ and use Lemmas 2.1 and 4.2 to obtain

$$\begin{aligned} \int_0^t \sigma_\tau(\tau) \|\boldsymbol{\rho}(\tau)\|^2 d\tau & \leq \int_0^t e^{2\alpha\tau} (\|\mathbf{e}^*(\tau)\|^2 + \|\boldsymbol{\zeta}(\tau)\|^2) d\tau \\ & \leq C \left((h^4 + h^2 \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^2) \int_0^T e^{2\alpha\tau} \mathcal{K}^2(\tau) d\tau + \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^2 \int_0^T e^{2\alpha\tau} \|\mathbf{e}_H(\tau)\|^2 d\tau \right). \end{aligned} \quad (4.43)$$

A use of Lemmas 3.1 and 4.2 yields

$$\begin{aligned} \int_0^t \sigma(\tau) b(\mathbf{e}_H(\tau), \mathbf{e}_H(\tau), \boldsymbol{\rho}) d\tau &\leq C \int_0^t \sigma(\tau) \|\mathbf{e}_H(\tau)\|^{1-\delta} \|\nabla \mathbf{e}_H(\tau)\|^{1+\delta} \|\nabla \boldsymbol{\rho}\| d\tau \\ &\leq C(\epsilon) \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{-2\delta} \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2(1+\delta)} \int_0^t \sigma(\tau) \|\mathbf{e}_H(\tau)\|^2 d\tau + \epsilon \int_0^t \sigma(\tau) \|\nabla \boldsymbol{\rho}(\tau)\|^2 d\tau. \end{aligned} \quad (4.44)$$

An application of (4.41)-(4.44) in (4.40) leads to

$$\|\boldsymbol{\rho}(t)\|^2 + \sigma^{-1}(t) \int_0^t \sigma(\tau) \|\nabla \boldsymbol{\rho}(\tau)\|^2 d\tau \leq K(t) (h^4 + H^{6-2\delta}). \quad (4.45)$$

A use of triangle inequality with (4.45) and Lemmas 4.3, 4.5 completes the proof of Theorem 4.1. For establishing uniform estimates in Theorem 4.1, use Lemma 4.2 in (4.42)-(4.44). \square

Next, we derive the error estimate for the two-grid approximation p_h^* of the pressure p . Now, consider an equivalent form of (3.4), that is, find $(\mathbf{u}_h^*(t), p_h^*(t)) \in \mathbf{H}_h \times L_h$ such that $\mathbf{u}_h^*(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\left. \begin{aligned} &(\mathbf{u}_{ht}^*, \boldsymbol{\phi}_h) + \nu a(\mathbf{u}_h^*, \boldsymbol{\phi}_h) + b(\mathbf{u}_h^*, \mathbf{u}_H, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{u}_h^*, \boldsymbol{\phi}_h) \\ &= (\mathbf{f}, \boldsymbol{\phi}_h) + b(\mathbf{u}_H, \mathbf{u}_H, \boldsymbol{\phi}_h) + (p_h^*, \nabla \cdot \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ &(\nabla \cdot \mathbf{u}_h^*, \chi_h) = 0 \quad \forall \chi_h \in L_h. \end{aligned} \right\} \quad (4.46)$$

To estimate $p - p_h^*$, use $j_h p$ and triangle inequality to obtain

$$\|p - p_h^*\| \leq \|p - j_h p\| + \|j_h p - p_h^*\|. \quad (4.47)$$

From (B2), observe that

$$\begin{aligned} \|j_h p - p_h^*\|_{L^2/N_h} &\leq C \sup_{\boldsymbol{\phi}_h \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(j_h p - p_h^*, \nabla \cdot \boldsymbol{\phi}_h)|}{\|\nabla \boldsymbol{\phi}_h\|} \right\} \\ &\leq C \left(\|j_h p - p\| + \sup_{\boldsymbol{\phi}_h \in \mathbf{H}_h \setminus \{0\}} \left\{ \frac{|(p - p_h^*, \nabla \cdot \boldsymbol{\phi}_h)|}{\|\nabla \boldsymbol{\phi}_h\|} \right\} \right). \end{aligned} \quad (4.48)$$

The first term on the right hand side of (4.48) can be estimated using (B1). To estimate the second term on the right hand side of (4.48), subtract (4.46) from (4.10) to obtain

$$\begin{aligned} (p - p_h^*, \nabla \cdot \boldsymbol{\phi}_h) &= (\mathbf{e}_t^*, \boldsymbol{\phi}_h) + \nu a(\mathbf{e}^*, \boldsymbol{\phi}_h) + b(\mathbf{e}^*, \mathbf{e}_H, \boldsymbol{\phi}_h) \\ &+ b(\mathbf{e}_H, \mathbf{e}^*, \boldsymbol{\phi}_h) - b(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\phi}_h) \quad \forall \boldsymbol{\phi}_h \in \mathbf{H}_h. \end{aligned} \quad (4.49)$$

A use of Lemma 3.1 yields

$$\begin{aligned} &|b(\mathbf{e}^*, \mathbf{e}_H, \boldsymbol{\phi}_h) + b(\mathbf{e}_H, \mathbf{e}^*, \boldsymbol{\phi}_h) - b(\mathbf{e}_H, \mathbf{e}_H, \boldsymbol{\phi}_h)| \\ &\leq C(\|\nabla \mathbf{e}_H\| \|\nabla \mathbf{e}^*\| + \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^{1+\delta}) \|\nabla \boldsymbol{\phi}_h\|. \end{aligned} \quad (4.50)$$

Apply the Cauchy-Schwarz's inequality with (4.50) to arrive at

$$(p - p_h^*, \nabla \cdot \boldsymbol{\phi}_h) \leq C(\|\mathbf{e}_t^*\|_{-1;h} + \|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{e}_H\| \|\nabla \mathbf{e}^*\| + \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^{1+\delta}) \|\nabla \boldsymbol{\phi}_h\|, \quad (4.51)$$

where

$$\|\mathbf{e}_t^*\|_{-1;h} = \sup \left\{ \frac{(\mathbf{e}_t^*, \boldsymbol{\phi}_h)}{\|\nabla \boldsymbol{\phi}_h\|} : \boldsymbol{\phi}_h \in \mathbf{H}_h, \boldsymbol{\phi}_h \neq 0 \right\}. \quad (4.52)$$

Since the estimate of $\|\nabla \mathbf{e}^*\|$ is known from Lemma 4.5, we now derive estimate of $\|\mathbf{e}_t^*\|_{-1;h}$. As $\mathbf{H}_h \subset \mathbf{H}_0^1$, we note that

$$\begin{aligned}\|\mathbf{e}_t^*\|_{-1;h} &= \sup \left\{ \frac{(\mathbf{e}_t^*, \phi_h)}{\|\nabla \phi_h\|} : \phi_h \in \mathbf{H}_h, \phi_h \neq 0 \right\} \\ &\leq \sup \left\{ \frac{(\mathbf{e}_t^*, \phi)}{\|\nabla \phi\|} : \phi \in \mathbf{H}_0^1, \phi \neq 0 \right\} = \|\mathbf{e}_t^*\|_{-1}.\end{aligned}\quad (4.53)$$

Lemma 4.7. *The error $\mathbf{e}^* = \mathbf{u} - \mathbf{u}_h^*$ satisfies for $0 < t < T$*

$$\|\mathbf{e}_t^*\|_{-1} \leq K(t) (h + H^{3-\delta}). \quad (4.54)$$

Proof. For any $\Psi \in \mathbf{H}_0^1$, use orthogogal projection $P_h : \mathbf{L}^2 \rightarrow \mathbf{J}_h$ and (4.11) with $\phi = P_h \Psi$ to obtain

$$\begin{aligned}(\mathbf{e}_t^*, \Psi) &= (\mathbf{e}_t^*, \Psi - P_h \Psi) + (\mathbf{e}_t^*, P_h \Psi) \\ &= (\mathbf{e}_t^*, \Psi - P_h \Psi) - \nu a(\mathbf{e}^*, P_h \Psi) - b(\mathbf{e}^*, \mathbf{u}_H, P_h \Psi) - b(\mathbf{u}_H, \mathbf{e}^*, P_h \Psi) \\ &\quad + b(\mathbf{e}_H, \mathbf{e}_H, P_h \Psi) + (p, \nabla \cdot P_h \Psi).\end{aligned}\quad (4.55)$$

Apply approximation property of P_h to find that

$$(\mathbf{e}_t^*, \Psi - P_h \Psi) = (\mathbf{u}_t, \Psi - P_h \Psi) \leq Ch \|\mathbf{u}_t\| \|\nabla P_h \Psi\|. \quad (4.56)$$

A use of the Cauchy-Schwarz's inequality with (4.50), (4.56), the discrete incompressibility condition and (4.53) in (4.55) leads to

$$\|\mathbf{e}_t^*\|_{-1} \leq C(h \|\mathbf{u}_t\| + \|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{e}_H\| \|\nabla \mathbf{e}^*\| + \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^{1+\delta} + h \|\nabla p\|). \quad (4.57)$$

Using Lemmas 2.1, 4.2 and 4.5, we arrive at the desired result. \square

Thus, we have proved the following theorem.

Theorem 4.2. *Under the hypotheses of Lemma 4.2, there exists a positive constant C depending on ν, γ, M_0 , such that, for all $t > 0$, it holds:*

$$\|(p - p_h^*)(t)\|_{L^2/N_h} \leq K(t)(h + H^{3-\delta}),$$

where $\delta > 0$ is arbitrarily small and $K(t) = Ce^{Ct}$. Under the uniqueness condition (3.7), $K(t) = C$ and the estimate is uniform in time.

4.2 Error estimates for Step 3

This section is devoted to the derivation of semidiscrete error estimates in **Step 3**.

Lemma 4.8. *Under the hypotheses of Lemma 4.4, the following estimate holds true:*

$$\tau(t) \|\mathbf{e}_t^*\|^2 + \beta \sigma^{-1}(t) \int_0^t \sigma_1(\tau) \|\nabla \mathbf{e}_\tau^*\|^2 \leq K(t) (h^2 + H^{6-2\delta}),$$

where $\sigma_1(t) = \tau^2(t)e^{2\alpha t}$ and $\beta = \gamma - \alpha \lambda_1^{-1} > 0$.

Proof. Differentiate (4.11) with respect to time and substitute $\phi_h = P_h \mathbf{e}_t^*$ in the resulting equation and use discrete incompressibility condition to arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{e}_t^*\|^2 + \gamma \|\nabla \mathbf{e}_t^*\|^2 &\leq (e_{tt}^*, \mathbf{u}_t - P_h \mathbf{u}_t) + \nu a(\mathbf{e}_t^*, \mathbf{u}_t - P_h \mathbf{u}_t) - b(\mathbf{e}_t^*, \mathbf{u}_H, \mathbf{u}_t - P_h \mathbf{u}_t) \\ &\quad - b(\mathbf{u}_H, \mathbf{e}_t^*, \mathbf{u}_t - P_h \mathbf{u}_t) - b(\mathbf{e}^*, \mathbf{u}_{Ht}, P_h \mathbf{e}_t^*) - b(\mathbf{u}_{Ht}, \mathbf{e}^*, P_h \mathbf{e}_t^*) \\ &\quad + b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h \mathbf{e}_t^*) + b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h \mathbf{e}_t^*) + (p_t - j_h p_t, \nabla \cdot P_h \mathbf{e}_t^*). \end{aligned} \quad (4.58)$$

A use of (3.1) with Lemma 3.1 and Theorem 4.2 yields

$$\begin{aligned} |b(\mathbf{e}_{Ht}, \mathbf{e}_H, P_h \mathbf{e}_t^*) + b(\mathbf{e}_H, \mathbf{e}_{Ht}, P_h \mathbf{e}_t^*)| &\leq C \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^\delta \|\nabla \mathbf{e}_{Ht}\| \|\nabla P_h \mathbf{e}_t^*\| \\ &\leq C(\epsilon) \|\mathbf{e}_H\|^{2(1-\delta)} \|\nabla \mathbf{e}_H\|^{2\delta} \|\nabla \mathbf{e}_{Ht}\|^2 + \epsilon \|\nabla \mathbf{e}_t^*\|^2. \end{aligned} \quad (4.59)$$

Apply (4.19)-(4.22) and (4.59) along with Cauchy-Schwarz's inequality in (4.58) to arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{e}_t^*\|^2 + \gamma \|\nabla \mathbf{e}_t^*\|^2 &\leq \frac{1}{2} \frac{d}{dt} \|\mathbf{u}_t - P_h \mathbf{u}_t\|^2 + C(\nu, \gamma) \left(h^2 \mathcal{K}_t^2 + h^2 \|\nabla \mathbf{u}_H\| \|\tilde{\Delta} \mathbf{u}_H\| \|\tilde{\Delta} \mathbf{u}_t\|^2 \right. \\ &\quad \left. + K(t) ((h^2 + H^{6-2\delta}) \|\nabla \mathbf{u}_{Ht}\|^2 + H^{4-2\delta} \|\nabla \mathbf{e}_{Ht}\|^2) \right). \end{aligned} \quad (4.60)$$

Use similar analysis to (4.60) as applied to (4.24) to arrive at (4.25) and Lemmas 2.1, 4.2, 4.5 to conclude the proof. \square

Lemma 4.9. *Under the hypotheses of Lemma 4.4, the following estimate holds true:*

$$\sigma^{-1}(t) \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}_h(\tau)\|^2 d\tau \leq K(t) (h^2 + H^{10-4\delta}).$$

Proof. Consider (4.12) with $\phi_h = P_h \mathbf{e}_h = (P_h \mathbf{u} - \mathbf{u}) + \mathbf{e}_h$. Then, use (3.6) to arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{e}_h\|^2 + \gamma \|\nabla \mathbf{e}_h\|^2 &\leq (\mathbf{e}_{ht}, \mathbf{u} - P_h \mathbf{u}) + \nu a(\mathbf{e}_h, \mathbf{u} - P_h \mathbf{u}) + b(\mathbf{u}_H, \mathbf{e}_h, \mathbf{u} - P_h \mathbf{u}) \\ &\quad + b(\mathbf{e}_h, \mathbf{u}_H, \mathbf{u} - P_h \mathbf{u}) - b(\mathbf{e}_H, \mathbf{e}^*, P_h \mathbf{e}_h) \\ &\quad - b(\mathbf{e}^*, \mathbf{e}_H, P_h \mathbf{e}_h) + b(\mathbf{e}^*, \mathbf{e}^*, P_h \mathbf{e}_h) + (p, \nabla \cdot P_h \mathbf{e}_h). \end{aligned} \quad (4.61)$$

The first four terms in the right hand side of (4.61) can be bounded using (4.19)-(4.21) (with \mathbf{e}^* replaced by \mathbf{e}_h). Now, from (3.1) and Lemma 3.1, we arrive at

$$\begin{aligned} |b(\mathbf{e}_H, \mathbf{e}^*, P_h \mathbf{e}_h) - b(\mathbf{e}^*, \mathbf{e}_H, P_h \mathbf{e}_h) + b(\mathbf{e}^*, \mathbf{e}^*, P_h \mathbf{e}_h)| \\ \leq C (\|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^\delta \|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{e}^*\|^2) \|\nabla \mathbf{e}_h\|. \end{aligned} \quad (4.62)$$

A use of (4.19)-(4.22) and (4.62) leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{e}_h\|^2 + \gamma \|\nabla \mathbf{e}_h\|^2 &\leq \frac{1}{2} \frac{d}{dt} \|\mathbf{u} - P_h \mathbf{u}\|^2 + C(\nu) h \mathcal{K} \|\nabla \mathbf{e}_h\| \\ &\quad + C(\nu) (\|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^\delta \|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{e}^*\|^2) \|\nabla \mathbf{e}_h\|. \end{aligned} \quad (4.63)$$

The proof can be concluded by using the similar set of arguments now to (4.63) as applied to (4.24) leading to (4.25) and Lemmas 4.2, 4.4 and 4.5. This completes the proof of Lemma 4.9. \square

Below, we state a lemma that provides $L^\infty(\mathbf{H}^1)$ -norm estimate for \mathbf{e}_h . The proof is obtained in the same lines as the proof of Lemma 4.5, starting with $\phi_h = P_h \mathbf{e}_{ht}$ in (4.12), using Lemmas 4.8 and 4.9 and hence, is skipped.

Lemma 4.10. *Under the hypotheses of Lemma 4.4, the following estimate holds true:*

$$\sigma^{-1}(t) \int_0^t \sigma(\tau) \|\mathbf{e}_{h\tau}(\tau)\|^2 d\tau + \|\nabla \mathbf{e}_h(t)\| \leq K(t) (h^2 + H^{8-2\delta}),$$

where $\delta > 0$ is arbitrarily small. \square

Lemma 4.11. *Under the hypotheses of Theorem 4.2, the following is satisfied:*

$$\sigma^{-1}(t) \int_0^t e^{2\alpha\tau} \|\mathbf{e}_h(\tau)\|^2 d\tau \leq K(t) (h^4 + h^2 H^4).$$

Proof. Proceeding in a similar way as in the proof of Lemma 4.6, we arrive at

$$\begin{aligned} \|\mathbf{e}_h\|^2 &= \frac{d}{dt} (\mathbf{e}_h, P_h \phi) - (\mathbf{u} - P_h \mathbf{u}, \phi_t) - a(\mathbf{e}_h, \phi - P_h \phi) - b(\mathbf{u}_H, \mathbf{e}_h, \phi - P_h \phi) \\ &\quad - b(\mathbf{e}_h, \mathbf{u}_H, \phi - P_h \phi) - b(\mathbf{e}_H, \mathbf{e}_h, \phi) - b(\mathbf{e}_h, \mathbf{e}_H, \phi) + b(\mathbf{e}_H, \mathbf{e}^*, P_h \phi - \phi) \\ &\quad + b(\mathbf{e}_H, \mathbf{e}^*, \phi) + b(\mathbf{e}^*, \mathbf{e}_H, P_h \phi - \phi) + b(\mathbf{e}^*, \mathbf{e}_H, \phi) - b(\mathbf{e}^*, \mathbf{e}^*, P_h \phi - \phi) \\ &\quad - b(\mathbf{e}^*, \mathbf{e}^*, \phi) - (p - j_h p, \nabla \cdot (P_h \phi - \phi)) + (\psi - j_h \psi, \nabla \cdot \mathbf{e}_h). \end{aligned} \quad (4.64)$$

Use (3.1), Lemmas 3.1 and 4.2 to bound

$$\begin{aligned} &|b(\mathbf{e}_H, \mathbf{e}^*, P_h \phi - \phi)| + |b(\mathbf{e}^*, \mathbf{e}_H, P_h \phi - \phi)| + |b(\mathbf{e}_H, \mathbf{e}^*, \phi)| + |b(\mathbf{e}_H, \mathbf{e}^*, \phi)| \\ &\leq C(h \|\mathbf{e}_H\|^{1-\delta} \|\nabla \mathbf{e}_H\|^\delta + \|\mathbf{e}_H\|) \|\nabla \mathbf{e}^*\| \|\phi\|_2 \leq K(t) (h H^{2-\delta} + H^2) \|\nabla \mathbf{e}^*\| \|\phi\|_2. \end{aligned} \quad (4.65)$$

An application of (3.1), Lemmas 3.1 and 4.5 yields

$$\begin{aligned} &|b(\mathbf{e}^*, \mathbf{e}^*, P_h \phi - \phi)| + |b(\mathbf{e}^*, \mathbf{e}^*, \phi)| \leq C(h \|\nabla \mathbf{e}^*\|^2 + \|\mathbf{e}^*\| \|\nabla \mathbf{e}^*\|) \|\phi\|_2 \\ &\leq K(t) (h^2 + H^{3-\delta}) \|\nabla \mathbf{e}^*\| \|\phi\|_2. \end{aligned} \quad (4.66)$$

Multiply (4.64) by $e^{2\alpha t}$ and integrate with respect to time from 0 to t . Then, apply (4.34)-(4.36) with \mathbf{e}^* replaced by \mathbf{e}_h and (4.65)-(4.66) to obtain

$$\begin{aligned} \int_0^t e^{2\alpha\tau} \|\mathbf{e}_h(\tau)\|^2 d\tau &\leq C(h^4 \int_0^t e^{2\alpha\tau} \mathcal{K}^2(\tau) d\tau + \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}_h(\tau)\|^2 (h^2 + \|\mathbf{e}_H(\tau)\|^2) d\tau) \\ &\quad + K(t) (h^2 + H^{4-2\delta}) e^{Ct} \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau \\ &\leq C(h^4 \int_0^t e^{2\alpha\tau} \mathcal{K}^2(\tau) d\tau + (h^2 + \|\mathbf{e}_H\|_{L^\infty(L^2)}^2) \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}_h(\tau)\|^2 d\tau) \\ &\quad + K(t) (h^2 + H^{4-2\delta}) \int_0^t e^{2\alpha\tau} \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau. \end{aligned} \quad (4.67)$$

A use of Lemmas 2.1, 4.2, 4.4 and 4.9 completes the rest part of the proof. \square

Proceeding in a similar way as in Lemma 4.7, we arrive at the following estimate.

Lemma 4.12. *The error $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$ satisfies for $0 < t < T$*

$$\|\mathbf{e}_{ht}\|_{-1} \leq K(t) (h + H^{4-\delta}).$$

For the pressure error estimates corresponding to the correction in **Step 3** of two-grid algorithm, consider the equivalent form of (3.5): seek $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{H}_h \times L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$\left. \begin{aligned} &(\mathbf{u}_{ht}, \phi_h) + \nu a(\mathbf{u}_h, \phi_h) + b(\mathbf{u}_h, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{u}_h, \phi_h) = (\mathbf{f}, \phi_h) \\ &\quad + b(\mathbf{u}_H, \mathbf{u}_h^*, \phi_h) + b(\mathbf{u}_h^*, \mathbf{u}_H - \mathbf{u}_h^*, \phi_h) + (p_h, \nabla \cdot \phi_h) \quad \forall \phi_h \in \mathbf{H}_h, \\ &(\nabla \cdot \mathbf{u}_h, \chi_h) = 0 \quad \forall \chi_h \in L_h. \end{aligned} \right\} \quad (4.68)$$

For pressure equation, subtract (4.68) from (4.10) to obtain

$$(p - p_h, \nabla \cdot \phi_h) = (\mathbf{e}_{ht}, \phi_h) + \nu a(\mathbf{e}_h, \phi_h) + b(\mathbf{e}_h, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{e}_h, \phi_h) - b(\mathbf{e}_H, \mathbf{e}^*, \phi_h) - b(\mathbf{e}^*, \mathbf{e}_H, \phi_h) - b(\mathbf{e}^*, \mathbf{e}^*, \phi_h) \quad \forall \phi_h \in \mathbf{H}_h. \quad (4.69)$$

Armed with these estimates, next we derive proof of main Theorem 3.1.

Proof of Theorem 3.1. Multiply (4.15) by $\sigma(t)$, substitute $\phi_h = \Theta$ and integrate the resulting equation from 0 to t to obtain

$$\begin{aligned} \sigma(t) \|\Theta(t)\|^2 + \gamma \int_0^t \sigma(\tau) \|\nabla \Theta(\tau)\|^2 d\tau &\leq - \int_0^t \sigma(\tau) (\zeta_\tau(\tau), \Theta) d\tau + \int_0^t \sigma_\tau(\tau) \|\Theta(\tau)\|^2 d\tau \\ &+ \int_0^t \sigma(\tau) (-b(\mathbf{e}_H(\tau), \mathbf{e}^*(\tau), \Theta) - b(\mathbf{e}^*(\tau), \mathbf{e}_H(\tau), \Theta) + b(\mathbf{e}^*(\tau), \mathbf{e}^*(\tau), \Theta)) d\tau. \end{aligned} \quad (4.70)$$

The first term on the right hand side of (4.70) can be tackled as in (4.41). Write $\Theta = \mathbf{e}_h - \zeta$ and use Lemmas 2.1, 4.3 to obtain

$$\int_0^t \sigma_\tau(\tau) \|\Theta(\tau)\|^2 d\tau \leq \int_0^t e^{2\alpha\tau} (\|\mathbf{e}_h(\tau)\|^2 + \|\zeta(\tau)\|^2) d\tau \leq K(t)(h^4 + h^2 H^{4-2\delta}) \sigma. \quad (4.71)$$

Use Young's inequality and Lemmas 3.1, 4.2, 4.4 to bound

$$\begin{aligned} & \left| \int_0^t \sigma(\tau) (b(\mathbf{e}_H(\tau), \mathbf{e}^*(\tau), \Theta) + b(\mathbf{e}^*(\tau), \mathbf{e}_H(\tau), \Theta)) d\tau \right| \\ & \leq C(\epsilon) \|\mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2-2\delta} \|\nabla \mathbf{e}_H\|_{L^\infty(\mathbf{L}^2)}^{2\delta} \int_0^t \sigma(\tau) \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau + \epsilon \int_0^t \sigma(\tau) \|\nabla \Theta(\tau)\|^2 d\tau \\ & \leq K(t)(h^2 H^{4-2\delta} + H^{10-4\delta}) \sigma + \epsilon \int_0^t \sigma(\tau) \|\nabla \Theta(\tau)\|^2 d\tau. \end{aligned} \quad (4.72)$$

An application of Lemmas 3.1, 4.4 and 4.5 leads to

$$\begin{aligned} \int_0^t \sigma(\tau) b(\mathbf{e}^*(\tau), \mathbf{e}^*(\tau), \Theta) d\tau &\leq C \int_0^t \sigma(\tau) \|\nabla \mathbf{e}^*(\tau)\|^2 \|\nabla \Theta\| d\tau \\ &\leq C \|\nabla \mathbf{e}^*\|_{L^\infty(\mathbf{L}^2)}^2 \int_0^t \sigma(\tau) \|\nabla \mathbf{e}^*(\tau)\|^2 d\tau + \epsilon \int_0^t \sigma(\tau) \|\nabla \Theta\|^2 d\tau \\ &\leq K(t)(h^4 + H^{12-4\delta}) \sigma + \epsilon \int_0^t \sigma(\tau) \|\nabla \Theta\|^2 d\tau. \end{aligned} \quad (4.73)$$

Apply Lemma 4.2 to obtain

$$\|\Theta(t)\|^2 + \sigma^{-1}(t) \int_0^t \sigma(\tau) \|\nabla \Theta(\tau)\|^2 d\tau \leq K(t)(h^4 + h^2 H^{4-2\delta} + H^{10-2\delta}). \quad (4.74)$$

A use of Lemmas 4.3, 4.10 with (4.74) completes the proof of Theorem 3.1.

The uniform estimates in Theorem 3.1 can be achieved by using Lemma 4.2 under uniqueness condition.

For the pressure estimate (3.9), a use of boundedness of $\|\nabla \mathbf{u}_H\|$ and Lemmas 3.1, 4.2, 4.5, 4.10 leads to

$$\begin{aligned} & |b(\mathbf{e}_h, \mathbf{u}_H, \phi_h) + b(\mathbf{u}_H, \mathbf{e}_h, \phi_h) - b(\mathbf{e}_H, \mathbf{e}^*, \phi_h) - b(\mathbf{e}^*, \mathbf{e}_H, \phi_h) - b(\mathbf{e}^*, \mathbf{e}^*, \phi_h)| \\ & \leq C(\|\nabla \mathbf{e}_h\| \|\nabla \mathbf{u}_H\| + \|\nabla \mathbf{e}_H\| \|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{e}^*\|^2) \|\nabla \phi_h\|. \end{aligned} \quad (4.75)$$

A use of Cauchy-Schwarz's inequality and (4.75) in (4.69) leads to

$$(p - p_h, \nabla \cdot \phi_h) \leq C (\|\mathbf{e}_{ht}\|_{-1} + \|\nabla \mathbf{e}_h\| + \|\nabla \mathbf{e}_h\| \|\nabla \mathbf{u}_H\| + \|\nabla \mathbf{e}_H\| \|\nabla \mathbf{e}^*\| + \|\nabla \mathbf{e}^*\|^2) \|\nabla \phi_h\|.$$

A use of Lemmas 4.2, 4.10, 4.12 completes the proof of the pressure estimate (3.9) and this concludes the rest of the proof of Theorem 3.1. \square

5 Backward Euler Method

For a complete discretization, we apply a backward Euler method for the time discretization. Let $\{t_n\}_{n=0}^N$ be a uniform partition of the time interval $[0, T]$ and $t_n = nk$, with time step $k > 0$. For a sequence $\{\phi^n\}_{n \geq 0} \in \mathbf{J}_h$ defined on $[0, T]$, set $\phi^n = \phi(t_n)$, $\bar{\partial}_t \phi^n = (\phi^n - \phi^{n-1})/k$.

The backward Euler method applied to (3.3)-(3.5) is stated in terms of the following algorithm:

Step 1. Solve the nonlinear system (1.1) on \mathcal{T}_H : find $\mathbf{U}_H^n \in \mathbf{J}_H$, such that for all $\phi_H \in \mathbf{J}_H$ for $\mathbf{U}_H^0 = P_H \mathbf{u}_0$ and $t > 0$

$$(\bar{\partial}_t \mathbf{U}_H^n, \phi_H) + \nu a(\mathbf{U}_H^n, \phi_H) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \phi_H) = (\mathbf{f}^n, \phi_H). \quad (5.1)$$

Step 2. Update on \mathcal{T}_h with one Newton iteration: find $\mathbf{U}^n \in \mathbf{J}_h$, such that for all $\phi_h \in \mathbf{J}_h$ for $\mathbf{U}^0 = P_h \mathbf{u}_0$ and $t > 0$

$$\begin{aligned} (\bar{\partial}_t \mathbf{U}^n, \phi_h) + \nu a(\mathbf{U}^n, \phi_h) + b(\mathbf{U}^n, \mathbf{U}_H^n, \phi_h) \\ + b(\mathbf{U}_H^n, \mathbf{U}^n, \phi_h) = (\mathbf{f}^n, \phi_h) + b(\mathbf{U}_H^n, \mathbf{U}_H^n, \phi_h). \end{aligned} \quad (5.2)$$

Step 3. Correct on \mathcal{T}_h : find $\mathbf{U}_h^n \in \mathbf{J}_h$ such that, for all $\phi_h \in \mathbf{J}_h$ for $\mathbf{U}_h^0 = P_h \mathbf{u}_0$ and $t > 0$

$$\begin{aligned} (\bar{\partial}_t \mathbf{U}_h^n, \phi_h) + \nu a(\mathbf{U}_h^n, \phi_h) + b(\mathbf{U}_h^n, \mathbf{U}_H^n, \phi_h) + b(\mathbf{U}_H^n, \mathbf{U}_h^n, \phi_h) \\ = (\mathbf{f}^n, \phi_h) + b(\mathbf{U}_H^n, \mathbf{U}^n, \phi_h) + b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \phi_h). \end{aligned} \quad (5.3)$$

The results in Lemmas 5.1-5.6 will play an important role in the derivation of error estimates in this section.

Lemma 5.1. Let \mathbf{u}_h^* be the solution of (3.4) on some interval $[0, T)$, $0 < T < \infty$ satisfying $\mathbf{u}_{0h}^* = P_h \mathbf{u}_0$. Then, there exists a positive constant $C = C(\gamma, \nu, \alpha, \lambda_1, M_0)$, such that for $0 \leq \alpha < \frac{\gamma \lambda_1}{2}$ for all $t > 0$, the following holds true:

$$\|\mathbf{u}_h^*(t)\|^2 + \|\nabla \mathbf{u}_h^*(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|\nabla \mathbf{u}_h^*(s)\|^2 + \|\tilde{\Delta} \mathbf{u}_h^*(s)\|^2) ds \leq C.$$

Proof. Multiply (3.4) by $e^{\alpha t}$ for some $\alpha > 0$ and set $\hat{\mathbf{u}}_h^* = e^{\alpha t} \mathbf{u}_h^*$. Substitute $\phi_h = \hat{\mathbf{u}}_h^*$ and use (2.4) ($\|\hat{\mathbf{u}}_h^*\|^2 \leq \lambda_1^{-1} \|\nabla \hat{\mathbf{u}}_h^*\|^2$) and (3.6) to obtain

$$\frac{1}{2} \frac{d}{dt} \|\hat{\mathbf{u}}_h^*\|^2 + \left(\gamma - \frac{\alpha}{\lambda_1} \right) \|\nabla \hat{\mathbf{u}}_h^*\|^2 \leq (\hat{\mathbf{f}}, \hat{\mathbf{u}}_h^*) + e^{-\alpha t} b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*). \quad (5.4)$$

An application of Cauchy-Schwarz's inequality and Young's inequality leads to

$$|(\hat{\mathbf{f}}, \hat{\mathbf{u}}_h^*)| \leq C(\lambda_1, \epsilon) \|\hat{\mathbf{f}}\|^2 + \epsilon \|\nabla \hat{\mathbf{u}}_h^*\|^2. \quad (5.5)$$

A use of Lemma 3.1, $\|\nabla \mathbf{u}_H\| \leq C$ and Young's inequality yields

$$\begin{aligned} |e^{-\alpha t} b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*)| &\leq C e^{-\alpha t} \|\nabla \hat{\mathbf{u}}_H\|^2 \|\nabla \hat{\mathbf{u}}_h^*\| \\ &\leq C(\epsilon) \|\nabla \hat{\mathbf{u}}_H\|^2 + \epsilon \|\nabla \hat{\mathbf{u}}_h^*\|^2. \end{aligned} \quad (5.6)$$

Apply (5.5)-(5.6) in (5.4) with $\epsilon = \frac{\gamma}{2}$ and integrate the resulting equation with respect to time to obtain

$$\|\hat{\mathbf{u}}_h^*(t)\|^2 + \left(\gamma - \frac{2\alpha}{\lambda_1} \right) \int_0^t \|\nabla \hat{\mathbf{u}}_h^*(s)\|^2 ds \leq \|\mathbf{u}_0\|^2 + C \int_0^t \left(\|\nabla \hat{\mathbf{u}}_H(s)\|^2 + \|\hat{\mathbf{f}}(s)\|^2 \right) ds.$$

Multiply above equation by $e^{-2\alpha t}$, use assumption **(A2)** and the fact that $e^{-2\alpha t} \int_0^t e^{2\alpha s} ds = \frac{1}{2\alpha}(1 - e^{-2\alpha t})$ to arrive at

$$\|\mathbf{u}_h^*(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla \mathbf{u}_h^*(s)\|^2 ds \leq C. \quad (5.7)$$

Next, multiply (3.4) by $e^{\alpha t}$ and rewrite it as

$$\begin{aligned} (\hat{\mathbf{u}}_{ht}^*, \phi_h) - \nu a(\tilde{\Delta}_h \hat{\mathbf{u}}_h^*, \phi_h) &= \alpha(\hat{\mathbf{u}}_h^*, \phi_h) - e^{-\alpha t} b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*, \phi_h) \\ &\quad - e^{-\alpha t} (b(\hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H, \phi_h) - b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \phi_h)) + (\hat{\mathbf{f}}, \phi_h). \end{aligned} \quad (5.8)$$

Substitute $\phi_h = -\tilde{\Delta}_h \hat{\mathbf{u}}_h^*$ in (5.8), note the fact that $-(\hat{\mathbf{u}}_{ht}^*, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*) = \frac{1}{2} \frac{d}{dt} \|\nabla \hat{\mathbf{u}}_h^*\|^2$ and integrate the resulting equation with respect to time to obtain

$$\begin{aligned} \|\nabla \hat{\mathbf{u}}_h^*(t)\|^2 + 2\nu \int_0^t \|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*(s)\|^2 ds &= \|\nabla \mathbf{u}_{0h}^*\|^2 - 2\alpha \int_0^t (\hat{\mathbf{u}}_h^*, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*) ds + 2 \int_0^t e^{-\alpha s} b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*) ds \\ &\quad + 2 \int_0^t e^{-\alpha s} \left(b(\hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*) - b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*) \right) ds - 2 \int_0^t (\hat{\mathbf{f}}, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*) ds. \end{aligned} \quad (5.9)$$

An application of Lemmas 3.1, 4.1 with Young's inequality yields

$$\begin{aligned} 2 \int_0^t e^{-\alpha s} \left(|b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_h^*, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*)| + |b(\hat{\mathbf{u}}_h^*, \hat{\mathbf{u}}_H, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*)| + |b(\hat{\mathbf{u}}_H, \hat{\mathbf{u}}_H, \tilde{\Delta}_h \hat{\mathbf{u}}_h^*)| \right) ds \\ \leq C(\epsilon) \int_0^t e^{-2\alpha s} (\|\tilde{\Delta}_h \hat{\mathbf{u}}_H(s)\|^2 + \|\nabla \hat{\mathbf{u}}_h^*(s)\|^2) ds + \epsilon \int_0^t \|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*(s)\|^2 ds. \end{aligned} \quad (5.10)$$

A use of (5.10) along with Cauchy-Schwarz's inequality leads to

$$\|\nabla \hat{\mathbf{u}}_h^*(t)\|^2 + \nu \int_0^t \|\tilde{\Delta}_h \hat{\mathbf{u}}_h^*(s)\|^2 ds \leq \|\nabla \mathbf{u}_{0h}^*\|^2 + C \int_0^t (\|\hat{\mathbf{f}}(s)\|^2 + \|\nabla \hat{\mathbf{u}}_h^*(s)\|^2 + \|\tilde{\Delta}_h \hat{\mathbf{u}}_H(s)\|^2) ds.$$

An application of (5.7), assumption **(A2)** and Lemma 4.1 completes the proof. \square

Lemma 5.2. *Under the assumption of Lemma 5.1, the following holds true:*

$$e^{-2\alpha t} \int_0^t e^{2\alpha s} (\|\mathbf{u}_{ht}^*(s)\|^2 + \|\mathbf{u}_{htt}^*(s)\|_{-1}^2) ds \leq C.$$

Proof. Substitute $\phi_h = e^{2\alpha t} \mathbf{u}_{ht}^*$ in (3.4) and write it as

$$\begin{aligned} e^{2\alpha t} \|\mathbf{u}_{ht}^*\|^2 &= \nu e^{2\alpha t} (\tilde{\Delta}_h \mathbf{u}_h^*, \mathbf{u}_{ht}^*) - e^{2\alpha t} (b(\mathbf{u}_H, \mathbf{u}_h^*, \mathbf{u}_{ht}^*) + b(\mathbf{u}_h^*, \mathbf{u}_H, \mathbf{u}_{ht}^*)) \\ &\quad + e^{2\alpha t} b(\mathbf{u}_H, \mathbf{u}_H, \mathbf{u}_{ht}^*) + e^{2\alpha t} (\mathbf{f}, \mathbf{u}_{ht}^*). \end{aligned} \quad (5.11)$$

Apply Lemmas 3.1, 4.1, 5.1 with Cauchy-Schwarz's inequality and Young's inequality to obtain

$$e^{2\alpha t} \|\mathbf{u}_{ht}^*\|^2 \leq C e^{2\alpha t} (\|\tilde{\Delta}_h \mathbf{u}_H\|^2 + \|\tilde{\Delta}_h \mathbf{u}_h^*\|^2 + \|\mathbf{f}\|^2). \quad (5.12)$$

Integrate (5.12) with respect to time and use Lemmas 4.1 and 5.1, assumption **(A2)** to arrive at

$$e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\mathbf{u}_{ht}^*(s)\|^2 ds \leq C. \quad (5.13)$$

Next, differentiate (3.4) with respect to time and obtain

$$\begin{aligned} & (\mathbf{u}_{htt}^*, \phi_h) + \nu a(\mathbf{u}_{ht}^*, \phi_h) + b(\mathbf{u}_H, \mathbf{u}_{ht}^*, \phi_h) + b(\mathbf{u}_{ht}^*, \mathbf{u}_H, \phi_h) = (\mathbf{f}_t, \phi_h) \\ & - (b(\mathbf{u}_{Ht}, \mathbf{u}_h^*, \phi_h) + b(\mathbf{u}_h^*, \mathbf{u}_{Ht}, \phi_h)) + (b(\mathbf{u}_H, \mathbf{u}_{Ht}, \phi_h) + b(\mathbf{u}_{Ht}, \mathbf{u}_H, \phi_h)). \end{aligned} \quad (5.14)$$

Substitute $\phi_h = e^{2\alpha t} \mathbf{u}_{ht}^*$ in (5.14) and use (3.6) to obtain

$$\frac{e^{2\alpha t}}{2} \frac{d}{dt} \|\mathbf{u}_{ht}^*\|^2 + \gamma e^{2\alpha t} \|\nabla \mathbf{u}_{ht}^*\|^2 \leq e^{2\alpha t} (\mathbf{f}_t, \mathbf{u}_{ht}^*) + I, \text{ say.} \quad (5.15)$$

A use of Lemma 3.1 yields

$$\begin{aligned} |I| & \leq e^{2\alpha t} |b(\mathbf{u}_{Ht}, \mathbf{u}_h^*, \mathbf{u}_{ht}^*)| + |b(\mathbf{u}_h^*, \mathbf{u}_{Ht}, \mathbf{u}_{ht}^*)| + |b(\mathbf{u}_{Ht}, \mathbf{u}_H, \mathbf{u}_{ht}^*)| + |b(\mathbf{u}_H, \mathbf{u}_{Ht}, \mathbf{u}_{ht}^*)| \\ & \leq C e^{2\alpha t} \|\mathbf{u}_{Ht}\| (\|\tilde{\Delta}_h \mathbf{u}_h^*\| + \|\tilde{\Delta}_H \mathbf{u}_H\|) \|\nabla \mathbf{u}_{ht}^*\|. \end{aligned} \quad (5.16)$$

Apply (2.4), (5.16), Cauchy-Schwarz's inequality and Young's inequality in (5.15) to arrive at

$$\frac{d}{dt} e^{2\alpha t} \|\mathbf{u}_{ht}^*\|^2 + \left(\gamma - \frac{2\alpha}{\lambda_1} \right) e^{2\alpha t} \|\nabla \mathbf{u}_{ht}^*\|^2 \leq C e^{2\alpha t} \left(\|\mathbf{f}_t\|^2 + \|\mathbf{u}_{Ht}\|^2 (\|\tilde{\Delta}_h \mathbf{u}_h^*\|^2 + \|\tilde{\Delta}_H \mathbf{u}_H\|^2) \right).$$

An integration with respect to time, a use of assumption **(A2)**, (5.12) and Lemmas 4.1, 5.1 leads to

$$\|\mathbf{u}_{ht}^*(t)\|^2 + e^{-2\alpha t} \int_0^t e^{2\alpha s} \|\nabla \mathbf{u}_{ht}^*(s)\|^2 ds \leq C. \quad (5.17)$$

Next, choose $\phi_h = -e^{2\alpha t} \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*$ in (5.14) and use Lemma 3.1 to arrive at

$$\begin{aligned} & |b(\mathbf{u}_{Ht}, \mathbf{u}_H, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*)| + |b(\mathbf{u}_H, \mathbf{u}_{Ht}, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*)| \leq C \|\nabla \mathbf{u}_H\| \|\nabla \mathbf{u}_{Ht}\| \|\mathbf{u}_{htt}^*\|_{-1}, \\ & |b(\mathbf{u}_{Ht}, \mathbf{u}_h^*, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*)| + |b(\mathbf{u}_h^*, \mathbf{u}_{Ht}, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*)| \leq C \|\nabla \mathbf{u}_{Ht}\| \|\nabla \mathbf{u}_h^*\| \|\mathbf{u}_{htt}^*\|_{-1}, \\ & |b(\mathbf{u}_H, \mathbf{u}_{ht}^*, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*)| + |b(\mathbf{u}_{ht}^*, \mathbf{u}_H, \tilde{\Delta}_h^{-1} \mathbf{u}_{htt}^*)| \leq C \|\nabla \mathbf{u}_{ht}^*\| \|\nabla \mathbf{u}_H\| \|\mathbf{u}_{htt}^*\|_{-1}. \end{aligned} \quad (5.18)$$

Integrate with respect to time from 0 to t , use (5.12), (5.18) and Lemma 5.1 to obtain

$$\begin{aligned} e^{2\alpha t} \|\mathbf{u}_{ht}^*(t)\|^2 + \int_0^t e^{2\alpha s} \|\mathbf{u}_{htt}^*(s)\|_{-1}^2 ds & \leq (\|\tilde{\Delta}_H \mathbf{u}_{0H}\|^2 + \|\tilde{\Delta}_h \mathbf{u}_{0h}^*\|^2) + C \left(\int_0^t e^{2\alpha s} \|\mathbf{u}_{ht}^*(s)\|^2 ds \right. \\ & \left. + \int_0^t e^{2\alpha s} (\|\nabla \mathbf{u}_{Ht}(s)\|^2 + \|\nabla \mathbf{u}_{ht}^*(s)\|^2) ds + \int_0^t e^{2\alpha s} \|\mathbf{f}_t(s)\|^2 ds \right). \end{aligned}$$

A use of (5.13), (5.17), **(A2)** and Lemma 4.1 concludes the proof of Lemma 5.2. \square

Lemma 5.3. (a priori bounds for \mathbf{U}_H^n) With $\alpha > 0$, choose k_0 small so that for $0 < k \leq k_0$,

$$1 + \left(\frac{\gamma \lambda_1}{2} \right) k \geq e^{\alpha k}. \quad (5.19)$$

Further, let $\mathbf{U}_H^0 = P_H \mathbf{u}_{0H}$. Then, discrete solution \mathbf{U}_H^n , $n \geq 1$ of (5.1) satisfies the following estimates:

$$\begin{aligned} \|\mathbf{U}_H^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \mathbf{U}_H^i\|^2 & \leq C(\gamma, \nu, \alpha, \lambda_1) (e^{-2\alpha t_n} \|\mathbf{U}_H^0\|^2 + \|\mathbf{f}\|_\infty^2), \\ \|\nabla \mathbf{U}_H^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\tilde{\Delta}_H \mathbf{U}_H^i\|^2 & \leq C(\gamma, \nu, \alpha, \lambda_1) (e^{-2\alpha t_n} \|\nabla \mathbf{U}_H^0\|^2 + \|\mathbf{f}\|_\infty^2), \end{aligned}$$

where $\|\mathbf{f}\|_\infty = \|\mathbf{f}\|_{L^\infty(\mathbf{L}^2)}$. \square

Lemma 5.4. (estimates for \mathbf{e}_H^n) Let the assumptions of Lemma 5.3 be satisfied. Also, let $u_H(t)$ be a solution of (3.3) and $\mathbf{e}_H^n = \mathbf{U}_H^n - \mathbf{u}_H^n$, for $n \geq 1$. Then, for some positive constant K_T , that depends on T , there holds

$$\|\mathbf{e}_H^n\|^2 + ke^{-2\alpha t_n} \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \mathbf{e}_H^i\|^2 \leq K_T k^2.$$

Lemma 5.5. (a priori bounds for \mathbf{U}^n) Under the hypotheses of Lemma 5.3, the discrete solution \mathbf{U}^n , $n \geq 1$ of (5.2) satisfies

$$\begin{aligned} \|\mathbf{U}^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \mathbf{U}^i\|^2 &\leq C(\gamma, \nu, \alpha, \lambda_1, T)(e^{-2\alpha t_n} \|\mathbf{U}^0\|^2 + \|\mathbf{f}\|_\infty^2). \\ \|\nabla \mathbf{U}^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\tilde{\Delta}_h \mathbf{U}^i\|^2 &\leq C(\gamma, \nu, \alpha, \lambda_1, T)(e^{-2\alpha t_n} \|\nabla \mathbf{U}^0\|^2 + \|\mathbf{f}\|_\infty^2). \end{aligned}$$

Proof. For $n = i$, multiply (5.2) by $e^{\alpha t_i}$, use $e^{\alpha t_i} \bar{\partial}_t \mathbf{U}^i = e^{\alpha k} \bar{\partial}_t \hat{\mathbf{U}}^i - \left(\frac{e^{\alpha k} - 1}{k}\right) \hat{\mathbf{U}}^i$ and divide the resulting equation by $e^{\alpha k}$ to obtain

$$\begin{aligned} (\bar{\partial}_t \hat{\mathbf{U}}^i, \phi_h) - \left(\frac{1 - e^{-\alpha k}}{k}\right) (\hat{\mathbf{U}}^i, \phi_h) + \nu e^{\alpha t_i} e^{-\alpha k} (a(\mathbf{U}^i, \phi_h) + b(\mathbf{u}_H^i, \mathbf{U}^i, \phi_h) + b(\mathbf{U}^i, \mathbf{u}_H^i, \phi_h)) \\ = e^{-\alpha k} (\hat{\mathbf{f}}^i, \phi_h) - e^{-\alpha t_{i+1}} b(\hat{\mathbf{e}}_H^i, \hat{\mathbf{U}}^i, \phi_h) - e^{-\alpha t_{i+1}} b(\hat{\mathbf{U}}^i, \hat{\mathbf{e}}_H^i, \phi_h) + e^{-\alpha t_{i+1}} b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}_H^i, \phi_h). \end{aligned} \quad (5.20)$$

Observe that

$$(\bar{\partial}_t \phi^i, \phi^i) = \frac{1}{2k} (\phi^i - \phi^{i-1}) \geq \frac{1}{2} \bar{\partial}_t \|\phi^i\|^2. \quad (5.21)$$

Substitute $\phi_h = \hat{\mathbf{U}}^i$ in (5.20), use (3.2) and (5.21) to arrive at

$$\begin{aligned} \frac{1}{2} \bar{\partial}_t \|\hat{\mathbf{U}}^i\|^2 + \left(\gamma e^{-\alpha k} - \left(\frac{1 - e^{-\alpha k}}{k}\right) \lambda_1^{-1}\right) \|\nabla \hat{\mathbf{U}}^i\|^2 \\ \leq e^{-\alpha k} (\hat{\mathbf{f}}^i, \hat{\mathbf{U}}^i) - e^{-\alpha t_{i+1}} b(\hat{\mathbf{U}}^i, \hat{\mathbf{e}}_H^i, \hat{\mathbf{U}}^i) + e^{-\alpha t_{i+1}} b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}^i). \end{aligned} \quad (5.22)$$

Applying (2.4) and Cauchy-Schwarz's inequality, the first term on the right hand side of (5.22) can be bounded as

$$|e^{-\alpha k} (\hat{\mathbf{f}}^i, \hat{\mathbf{U}}^i)| \leq C e^{-\alpha k} \|\hat{\mathbf{f}}^i\| \|\hat{\mathbf{U}}^i\| \leq C(\lambda_1, \epsilon) e^{-\alpha k} \|\hat{\mathbf{f}}^i\|^2 + \epsilon e^{-\alpha k} \|\nabla \hat{\mathbf{U}}^i\|^2. \quad (5.23)$$

A use of Lemma 3.1 with Young's inequality yields

$$\begin{aligned} e^{-\alpha t_{i+1}} (|b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}^i)| + |b(\hat{\mathbf{U}}^i, \hat{\mathbf{e}}_H^i, \hat{\mathbf{U}}^i)|) &\leq C(\epsilon) e^{-2\alpha t_i} e^{-\alpha k} (\|\nabla \hat{\mathbf{U}}_H^i\|^4 + \|\nabla \hat{\mathbf{e}}_H^i\|^2 \|\hat{\mathbf{U}}^i\|^2) \\ &\quad + \epsilon e^{-\alpha k} \|\nabla \hat{\mathbf{U}}^i\|^2. \end{aligned} \quad (5.24)$$

Apply (5.23)-(5.24) with $\epsilon = \gamma/2$ in (5.22) to obtain

$$\begin{aligned} \bar{\partial}_t \|\hat{\mathbf{U}}^i\|^2 + \left(\gamma e^{-\alpha k} - 2 \left(\frac{1 - e^{-\alpha k}}{k}\right) \lambda_1^{-1}\right) \|\nabla \hat{\mathbf{U}}^i\|^2 \\ \leq C e^{-\alpha k} \left(\|\hat{\mathbf{f}}^i\|^2 + e^{-2\alpha t_i} \|\nabla \hat{\mathbf{U}}_H^i\|^4 + e^{-2\alpha t_i} \|\nabla \hat{\mathbf{e}}_H^i\|^2 \|\hat{\mathbf{U}}^i\|^2\right). \end{aligned} \quad (5.25)$$

We choose $k_0 > 0$, such that $1 + \left(\frac{\gamma\lambda_1}{2}\right) k \geq e^{\alpha k}$. This guarantees that $\gamma e^{-\alpha k} - 2 \left(\frac{1-e^{-\alpha k}}{k}\right) \lambda_1^{-1} \geq 0$. Multiply (5.25) by k and then sum over $i = 1$ to n to obtain

$$\begin{aligned} \|\hat{\mathbf{U}}^n\|^2 + \left(\gamma e^{-\alpha k} - 2 \left(\frac{1-e^{-\alpha k}}{k}\right) \lambda_1^{-1}\right) k \sum_{i=1}^n \|\nabla \hat{\mathbf{U}}^i\|^2 &\leq \|\mathbf{U}^0\|^2 + C \left(k e^{-\alpha k} \|\mathbf{f}\|_\infty^2 \sum_{i=1}^n e^{2\alpha t_i} \right. \\ &\quad \left. + k \sum_{i=1}^n e^{-2\alpha t_i} \|\nabla \hat{\mathbf{U}}_H^i\|^4 + k \sum_{i=1}^n e^{-2\alpha t_i} \|\nabla \hat{\mathbf{e}}_H^i\|^2 \|\hat{\mathbf{U}}^i\|^2 \right). \end{aligned}$$

An application of Gronwall's lemma with Lemmas 5.3 and 5.4 leads to the desired result. For $n = i$, multiply (5.2) by $e^{2\alpha t_i}$ and choose $\phi_h = -\tilde{\Delta}_h \hat{\mathbf{U}}^i$ to arrive at

$$\begin{aligned} \frac{1}{2} \bar{\partial}_t \|\nabla \hat{\mathbf{U}}^i\|^2 + \nu e^{-\alpha k} \|\tilde{\Delta}_h \hat{\mathbf{U}}^i\|^2 &\leq -e^{-\alpha k} (\hat{\mathbf{f}}^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i) - \left(\frac{1-e^{-\alpha k}}{k}\right) (\hat{\mathbf{U}}^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i) \\ &\quad + e^{-\alpha t_{i+1}} (b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i) + b(\hat{\mathbf{U}}^i, \hat{\mathbf{U}}_H^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i)) - e^{-\alpha t_{i+1}} b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}_H^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i). \end{aligned} \quad (5.26)$$

A use of Lemma 3.1 leads to

$$\begin{aligned} |b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i)| + |b(\hat{\mathbf{U}}^i, \hat{\mathbf{U}}_H^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i)| + |b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}_H^i, \tilde{\Delta}_h \hat{\mathbf{U}}^i)| \\ \leq C(\|\tilde{\Delta}_H \hat{\mathbf{U}}_H^i\| \|\nabla \hat{\mathbf{U}}^i\| + \|\nabla \hat{\mathbf{U}}_H^i\| \|\tilde{\Delta}_H \hat{\mathbf{U}}_H^i\|) \|\tilde{\Delta}_h \hat{\mathbf{U}}^i\|. \end{aligned} \quad (5.27)$$

Multiply (5.26) by k and then sum over $i = 1$ to n and use (2.4), (5.27) to arrive at

$$\begin{aligned} \|\nabla \hat{\mathbf{U}}^n\|^2 + \nu e^{-\alpha k} k \sum_{i=1}^n \|\tilde{\Delta}_h \hat{\mathbf{U}}^i\|^2 &\leq \|\nabla \mathbf{U}^0\|^2 + C(\lambda_1) k \sum_{i=1}^n e^{-\alpha k} (\|\hat{\mathbf{f}}^i\|^2 + \|\nabla \hat{\mathbf{U}}^i\|^2) \\ &\quad + e^{-\alpha k} k \sum_{i=1}^n e^{-2\alpha t_i} (\|\tilde{\Delta}_H \hat{\mathbf{U}}_H^i\|^2 \|\nabla \hat{\mathbf{U}}^i\|^2 + \|\nabla \hat{\mathbf{U}}_H^i\|^2 \|\tilde{\Delta}_H \hat{\mathbf{U}}_H^i\|^2). \end{aligned} \quad (5.28)$$

An application of Gronwall's lemma with **(A2)**, Lemmas 5.3 and 5.5 concludes the proof. \square

Lemma 5.6. *(a priori bounds for \mathbf{U}_h^n) Under the hypotheses of Lemma 5.3, the discrete solution \mathbf{U}_h^n , $n \geq 1$ of (5.3) satisfies*

$$\begin{aligned} \|\mathbf{U}_h^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \mathbf{U}_h^i\|^2 &\leq C(\gamma, \nu, \alpha, \lambda_1, T) (e^{-2\alpha t_n} \|\mathbf{U}^0\|^2 + \|\mathbf{f}\|_\infty^2), \\ \|\nabla \mathbf{U}_h^n\|^2 + e^{-2\alpha t_n} k \sum_{i=1}^n e^{2\alpha t_i} \|\tilde{\Delta}_h \mathbf{U}_h^i\|^2 &\leq C(\gamma, \nu, \alpha, \lambda_1, T) (e^{-2\alpha t_n} \|\nabla \mathbf{U}^0\|^2 + \|\mathbf{f}\|_\infty^2). \end{aligned}$$

Proof. For $n = i$, multiply (5.3) by $e^{\alpha t_i}$, substitute $\phi_h = \hat{\mathbf{U}}_h^i$, use (3.2) and (5.21) to arrive at

$$\begin{aligned} \frac{1}{2} \bar{\partial}_t \|\hat{\mathbf{U}}_h^i\|^2 + \left(\gamma e^{-\alpha k} - \left(\frac{1-e^{-\alpha k}}{k}\right) \lambda_1^{-1}\right) \|\nabla \hat{\mathbf{U}}_h^i\|^2 &\leq e^{-\alpha k} (\hat{\mathbf{f}}^i, \hat{\mathbf{U}}_h^i) \\ &\quad - e^{-\alpha t_{i+1}} b(\hat{\mathbf{U}}_h^i, \hat{\mathbf{e}}_H^i, \hat{\mathbf{U}}_h^i) + e^{-\alpha t_{i+1}} \left(b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}^i, \hat{\mathbf{U}}_h^i) + b(\hat{\mathbf{U}}^i, \hat{\mathbf{U}}_H^i - \hat{\mathbf{U}}^i, \hat{\mathbf{U}}_h^i)\right). \end{aligned} \quad (5.29)$$

The first two terms on the right hand side of (5.29) can be tackled similar to (5.23)-(5.24). To bound the third term, use Lemmas 3.1, 5.3, 5.5 and Young's inequality and arrive at

$$\begin{aligned} e^{-\alpha t_{i+1}} |b(\hat{\mathbf{U}}_H^i, \hat{\mathbf{U}}^i, \hat{\mathbf{U}}_h^i) + b(\hat{\mathbf{U}}^i, \hat{\mathbf{U}}_H^i - \hat{\mathbf{U}}^i, \hat{\mathbf{U}}_h^i)| &\leq C e^{-2\alpha t_i} e^{-\alpha k} (\|\nabla \hat{\mathbf{U}}_H^i\|^2 + \|\nabla \hat{\mathbf{U}}^i\|^2) \\ &\quad + \epsilon e^{-\alpha k} \|\nabla \hat{\mathbf{U}}_h^i\|^2. \end{aligned} \quad (5.30)$$

A use of (5.23)-(5.24) and (5.30) in (5.29) yields

$$\begin{aligned} \bar{\partial}_t \|\hat{\mathbf{U}}_h^i\|^2 + \left(\gamma e^{-\alpha k} - 2 \left(\frac{1 - e^{-\alpha k}}{k} \right) \lambda_1^{-1} \right) \|\nabla \hat{\mathbf{U}}_h^i\|^2 &\leq C e^{-\alpha k} \times \\ \left(\|\hat{\mathbf{f}}^i\|^2 + e^{-2\alpha t_i} (\|\nabla \hat{\mathbf{U}}_H^i\|^2 + \|\nabla \hat{\mathbf{U}}^i\|^2) + e^{-2\alpha t_i} \|\nabla \hat{\mathbf{e}}_H^i\|^2 \|\hat{\mathbf{U}}_h^i\|^2 \right). \end{aligned} \quad (5.31)$$

Multiply (5.31) by k , sum over $i = 1$ to n and use Lemmas 5.3-5.5 to complete the proof. \square

5.1 A Priori Error Estimates

Consider (3.3)-(3.5) at $t = t_n$ and subtract the resulting equations from (5.1)-(5.3), respectively, to arrive at the following error equations:

Step 1. for all $\phi_H \in \mathbf{J}_H$

$$(\bar{\partial}_t \mathbf{e}_H^n, \phi_H) + \nu a(\mathbf{e}_H^n, \phi_H) + b(\mathbf{u}_H^n, \mathbf{e}_H^n, \phi_H) + b(\mathbf{e}_H^n, \mathbf{u}_H^n, \phi_H) = (\sigma_H^n, \phi_H) + \Lambda_H(\phi_H), \quad (5.32)$$

where $\Lambda_H(\phi_H) = b(\mathbf{u}_H^n, \mathbf{e}_H^n, \phi_H) - b(\mathbf{U}_H^n, \mathbf{e}_H^n, \phi_H)$ and $\sigma_H^n = \mathbf{u}_{Ht}^n - \bar{\partial}_t \mathbf{u}_H^n$.

Step 2. for all $\phi_h \in \mathbf{J}_h$

$$(\bar{\partial}_t \mathbf{e}^n, \phi_h) + \nu a(\mathbf{e}^n, \phi_h) + b(\mathbf{u}_H^n, \mathbf{e}^n, \phi_h) + b(\mathbf{e}^n, \mathbf{u}_H^n, \phi_h) = (\sigma^n, \phi_h) + \Lambda^*(\phi_h), \quad (5.33)$$

where $\sigma^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^{*n}$, $\Lambda^*(\phi_h) = \Lambda_1(\phi_h) + \Lambda_2(\phi_h) + \Lambda_3(\phi_h)$ with

$$\left. \begin{aligned} \Lambda_1(\phi_h) &= -b(\mathbf{e}_H^n, \mathbf{U}^n, \phi_h), \\ \Lambda_2(\phi_h) &= -b(\mathbf{U}^n, \mathbf{e}_H^n, \phi_h), \\ \Lambda_3(\phi_h) &= b(\mathbf{U}_H^n, \mathbf{U}_H^n, \phi_h) - b(\mathbf{u}_H^n, \mathbf{u}_H^n, \phi_h). \end{aligned} \right\} \quad (5.34)$$

Step 3. for all $\phi_h \in \mathbf{J}_h$

$$(\bar{\partial}_t \mathbf{e}_h^n, \phi_h) + \nu a(\mathbf{e}_h^n, \phi_h) + b(\mathbf{u}_H^n, \mathbf{e}_h^n, \phi_h) + b(\mathbf{e}_h^n, \mathbf{u}_H^n, \phi_h) = (\sigma_h^n, \phi_h) + \Lambda_h(\phi_h), \quad (5.35)$$

where $\sigma_h^n = \mathbf{u}_{ht}^n - \bar{\partial}_t \mathbf{u}_h^n$, $\Lambda_h(\phi_h) = \Lambda_h^1(\phi_h) + \Lambda_h^2(\phi_h) + \Lambda_h^3(\phi_h) + \Lambda_h^4(\phi_h)$ with

$$\left. \begin{aligned} \Lambda_h^1(\phi_h) &= -b(\mathbf{e}_H^n, \mathbf{U}_h^n, \phi_h), \\ \Lambda_h^2(\phi_h) &= -b(\mathbf{U}_h^n, \mathbf{e}_H^n, \phi_h), \\ \Lambda_h^3(\phi_h) &= b(\mathbf{U}_H^n, \mathbf{U}^n, \phi_h) - b(\mathbf{u}_H^n, \mathbf{u}_h^{*n}, \phi_h), \\ \Lambda_h^4(\phi_h) &= b(\mathbf{U}^n, \mathbf{U}_H^n - \mathbf{U}^n, \phi_h) - b(\mathbf{u}_h^{*n}, \mathbf{u}_H^n - \mathbf{u}_h^{*n}, \phi_h). \end{aligned} \right\} \quad (5.36)$$

The main result of this section is stated as:

Theorem 5.1. (fully discrete error estimates) Under the assumptions of Theorem 3.1 and Lemma 5.3, the following hold true:

$$\begin{aligned} \|\mathbf{u}(t_n) - \mathbf{U}_h^n\| &\leq C(h^2 + H^{4-2\delta} + k), & \|\nabla(\mathbf{u}(t_n) - \mathbf{U}_h^n)\| &\leq C(h + H^{4-\delta} + k), \\ \|p(t_n) - P_h^n\| &\leq C(h + H^{4-\delta} + k^{1/2}), \end{aligned}$$

where $\delta > 0$ is arbitrarily small.

Below, we prove a lemma which will be used subsequently.

Lemma 5.7. Assume that (A1)-(A2) and (B1)-(B2) hold true. Let for some fixed h , \mathbf{u}_h^* satisfies (3.4). Then, there is a positive constant K_T that depends on T such that

$$\|\mathbf{e}^i\|^2 + k e^{-2\alpha t_n} \sum_{i=1}^n e^{2\alpha t_i} \|\nabla \mathbf{e}^i\|^2 \leq K_T k^2.$$

Proof. For $n = i$, substitute $\phi_h = \mathbf{e}^i$ in (5.33) and use (5.21) to obtain

$$\bar{\partial}_t \|\mathbf{e}^i\|^2 + 2\gamma \|\nabla \mathbf{e}^i\|^2 \leq 2(\sigma^i, \mathbf{e}^i) + 2\Lambda^*(\mathbf{e}^i). \quad (5.37)$$

Multiply (5.37) by $e^{2\alpha ik}$ and sum over $i = 1$ to n , where $T = nk$. Use the fact

$$\begin{aligned} \sum_{i=1}^n k e^{2\alpha ik} \bar{\partial}_t \|\mathbf{e}^i\|^2 &= \sum_{i=1}^n e^{2\alpha ik} (\|\mathbf{e}^i\|^2 - \|\mathbf{e}^{i-1}\|^2) \\ &= e^{2\alpha nk} \|\mathbf{e}^n\|^2 - \sum_{i=1}^{n-1} e^{2\alpha ik} (e^{2\alpha k} - 1) \|\mathbf{e}^i\|^2 \end{aligned} \quad (5.38)$$

to arrive at

$$\begin{aligned} e^{2\alpha nk} \|\mathbf{e}^n\|^2 + 2k\gamma \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}^i\|^2 &\leq \sum_{i=1}^{n-1} e^{2\alpha ik} (e^{2\alpha k} - 1) \|\mathbf{e}^i\|^2 \\ &\quad + 2k \sum_{i=1}^n e^{2\alpha ik} (\sigma^i, \mathbf{e}^i) + 2k \sum_{i=1}^n e^{2\alpha ik} \Lambda^*(\mathbf{e}^i). \end{aligned} \quad (5.39)$$

A use of Taylor's series expansion in the interval (t_{i-1}, t_i) with use of Cauchy-Schwarz's and Young's inequalities yields

$$\begin{aligned} |2(\sigma^i, \mathbf{e}^i)| &\leq \frac{2}{k} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \|\mathbf{u}_{htt}^*\|_{-1} dt \|\nabla \mathbf{e}^i\| \\ &\leq K k^{1/2} \left\{ \int_{t_{i-1}}^{t_i} \|\mathbf{u}_{htt}^*\|_{-1}^2 dt \right\}^{1/2} \|\nabla \mathbf{e}^i\|. \end{aligned} \quad (5.40)$$

From Lemma 5.2, observe that

$$\begin{aligned} \sum_{i=1}^n e^{2\alpha ik} \int_{t_{i-1}}^{t_i} \|\mathbf{u}_{htt}^*\|_{-1}^2 dt &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} e^{2\alpha(t_i-t)} e^{2\alpha t} \|\mathbf{u}_{htt}^*\|_{-1}^2 dt \\ &\leq e^{2\alpha k} \int_0^{t_n} e^{2\alpha t} \|\mathbf{u}_{htt}^*\|_{-1}^2 dt \leq K e^{2\alpha(n+1)k}. \end{aligned} \quad (5.41)$$

Apply (5.34) to obtain

$$|2k \sum_{i=1}^n e^{2\alpha ik} \Lambda^*(\mathbf{e}^i)| \leq 2k \sum_{i=1}^n e^{2\alpha ik} (|\Lambda_1(\mathbf{e}^i)| + |\Lambda_2(\mathbf{e}^i)| + |\Lambda_3(\mathbf{e}^i)|). \quad (5.42)$$

An application of Lemma 3.1 with Young's inequality and Lemmas 5.4 and 5.5 leads to

$$\begin{aligned} 2k \sum_{i=1}^n e^{2\alpha ik} (|\Lambda_1(\mathbf{e}^i)| + |\Lambda_2(\mathbf{e}^i)|) &\leq Ck \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_H^i\| \|\nabla \mathbf{U}^i\| \|\nabla \mathbf{e}^i\| \\ &\leq C(\epsilon)k \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_H^i\|^2 \|\nabla \mathbf{U}^i\|^2 + \epsilon k \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}^i\|^2 \\ &\leq C(T, \epsilon)k^2 e^{2\alpha nk} + \epsilon k \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}^i\|^2. \end{aligned} \quad (5.43)$$

A use of boundedness of $\|\nabla \mathbf{u}_H\| \leq C$ and Lemmas 3.1 and 5.3 to obtain

$$\begin{aligned} |\Lambda_3(\mathbf{e}^i)| &= |b(\mathbf{e}_H^i, \mathbf{u}_H^i, \mathbf{e}^i) + b(\mathbf{U}_H^i, \mathbf{e}_H^i, \mathbf{e}^i)| \\ &\leq C(\|\nabla \mathbf{U}_H^i\| + \|\nabla \mathbf{u}_H^i\|) \|\nabla \mathbf{e}_H^i\| \|\nabla \mathbf{e}^i\| \leq C \|\nabla \mathbf{e}_H^i\| \|\nabla \mathbf{e}^i\|. \end{aligned}$$

Now, a use of Young's inequality and Lemma 5.4 yields

$$\begin{aligned} |2k \sum_{i=1}^n e^{2\alpha i k} \Lambda_3(\mathbf{e}^i)| &\leq C(\epsilon) k \sum_{i=1}^n e^{2\alpha i k} \|\nabla \mathbf{e}_H^i\|^2 + \epsilon k \sum_{i=1}^n e^{2\alpha i k} \|\nabla \mathbf{e}^i\|^2 \\ &\leq C(T, \epsilon) k^2 e^{2\alpha n k} + \epsilon k \sum_{i=1}^n e^{2\alpha i k} \|\nabla \mathbf{e}^i\|^2. \end{aligned} \quad (5.44)$$

With the help of (5.40)-(5.44), (5.39) can be written as

$$e^{2\alpha n k} \|\mathbf{e}^n\|^2 + k \sum_{i=1}^n e^{2\alpha i k} \|\nabla \mathbf{e}^i\|^2 \leq C k^2 (e^{2\alpha(n+1)k} + e^{2\alpha n k}) + C k \sum_{i=1}^{n-1} e^{2\alpha i k} \|\mathbf{e}^i\|^2.$$

A use of discrete Gronwall's Lemma leads to

$$\|\mathbf{e}^n\|^2 + k e^{-2\alpha n k} \sum_{i=1}^n e^{2\alpha i k} \|\nabla \mathbf{e}^i\|^2 \leq K_T k^2$$

and this completes the rest of the proof. \square

Now, substitute $\phi_h = (-\tilde{\Delta}_h)^{-1} \bar{\partial}_t \mathbf{e}^n$ in (5.33) and use Lemma 3.1 to arrive at

$$\begin{aligned} \|\bar{\partial}_t \mathbf{e}^n\|_{-1}^2 &\leq C(\nu) (\|\nabla \mathbf{e}^n\| + \|\sigma^n\| + \|\nabla \mathbf{u}_H^n\| \|\nabla \mathbf{e}^n\| \\ &\quad + (\|\nabla \mathbf{U}_H^n\| + \|\nabla \mathbf{u}_H^n\| + \|\nabla \mathbf{U}^n\|) \|\nabla \mathbf{e}_H^n\|) \|\bar{\partial}_t \mathbf{e}^n\|_{-1}. \end{aligned} \quad (5.45)$$

An application of (5.40) and Lemmas 4.1, 5.2, 5.3, 5.4, 5.5, 5.7 leads to

$$\|\bar{\partial}_t \mathbf{e}^n\|_{-1}^2 \leq C k. \quad (5.46)$$

Next, to derive pressure error estimates, we consider the equivalent form of semidiscrete approximations (3.4) as: find $(\mathbf{u}_h^*(t), p_h^*(t)) \in \mathbf{H}_h \times L_h$ such that $\mathbf{u}_h^*(0) = \mathbf{u}_{0h}$ and for $t > 0$,

$$\left. \begin{aligned} &(\mathbf{u}_{ht}^*, \phi_h) + \nu a(\mathbf{u}_h^*, \phi_h) + b(\mathbf{u}_h^*, \mathbf{u}_H, \phi_h) \\ &+ b(\mathbf{u}_H, \mathbf{u}_h^*, \phi_h) = b(\mathbf{u}_H, \mathbf{u}_H, \phi_h) + (p_h^*, \nabla \cdot \phi_h) \quad \forall \phi_h \in \mathbf{H}_h, \\ &(\nabla \cdot \mathbf{u}_h^*, \chi_h) = 0 \quad \forall \chi_h \in L_h. \end{aligned} \right\} \quad (5.47)$$

The equivalent form of fully discrete approximations (5.2) is as follows: $\forall (\phi_h, \chi_h) \in \mathbf{H}_h \times L_h$, seek a sequence of functions $(\mathbf{U}^n, P^n)_{n \geq 1} \in \mathbf{H}_h \times L_h$ as solutions of the following equations:

$$\left. \begin{aligned} &(\bar{\partial}_t \mathbf{U}^n, \phi_h) + \nu a(\mathbf{U}^n, \phi_h) + b(\mathbf{U}^n, \mathbf{U}_H^n, \phi_h) \\ &+ b(\mathbf{U}_H^n, \mathbf{U}^n, \phi_h) = b(\mathbf{U}_H^n, \mathbf{U}_H^n, \phi_h) + (P^n, \nabla \cdot \phi_h), \\ &(\nabla \cdot \mathbf{U}^n, \chi_h) = 0. \end{aligned} \right\} \quad (5.48)$$

Subtract (5.47) from (5.48) and write $\rho^n = P^n - p_h^{*n}$ to obtain

$$(\rho^n, \nabla \cdot \phi_h) = (\bar{\partial}_t \mathbf{e}^n, \phi_h) + \nu a(\mathbf{e}^n, \phi_h) - \Lambda^*(\phi_h) - (\sigma^n, \phi_h).$$

A use of the Cauchy-Schwarz's inequality along with (5.40), (5.46) and Lemmas 4.1, 5.2, 5.3, 5.4, 5.5, 5.7 yields

$$\|\rho^n\| \leq C(\kappa, \nu, \lambda_1, M) k^{1/2}. \quad (5.49)$$

A combination of (5.49) and Theorem 4.2 leads to the following pressure estimate.

$$\|p(t_n) - P^n\| \leq C(h + H^{3-\delta} + k^{1/2}).$$

Proof of Theorem 5.1. Write $\mathbf{u}(t_n) - \mathbf{U}_h^n = (\mathbf{u}(t_n) - \mathbf{u}_h(t_n)) - \mathbf{e}_h^n$. The estimate of $\mathbf{u}(t_n) - \mathbf{u}_h(t_n)$ is obtained in Theorem 3.1. Next, we proceed to derive the estimates for \mathbf{e}_h^n . For $n = i$, substitute $\phi_h = \mathbf{e}_h^i$ in (5.35) and use (5.21). Multiply the resulting equation by $e^{2\alpha ik}$ and sum over $i = 1$ to n to obtain

$$\begin{aligned} e^{2\alpha nk} \|\mathbf{e}_h^n\|^2 + 2k\gamma \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_h^i\|^2 &\leq \sum_{i=1}^{n-1} e^{2\alpha ik} (e^{2\alpha k} - 1) \|\mathbf{e}_h^i\|^2 \\ &\quad + 2k \sum_{i=1}^n e^{2\alpha ik} (\sigma_h^i, \mathbf{e}_h^i) + 2k \sum_{i=1}^n e^{2\alpha ik} \Lambda_h(\mathbf{e}_h^i). \end{aligned} \quad (5.50)$$

The second term in the right hand side of (5.50) can be bounded similar to (5.40)-(5.41). Also, from (5.36) observe that

$$2k \sum_{i=1}^n e^{2\alpha ik} |\Lambda_h(\mathbf{e}_h^i)| \leq 2k \sum_{i=1}^n e^{2\alpha ik} (|\Lambda_h^1(\mathbf{e}_h^i)| + |\Lambda_h^2(\mathbf{e}_h^i)| + |\Lambda_h^3(\mathbf{e}_h^i)| + |\Lambda_h^4(\mathbf{e}_h^i)|). \quad (5.51)$$

An application of Lemmas 3.1, 5.4 and 5.6 yields

$$2k \sum_{i=1}^n e^{2\alpha ik} (|\Lambda_h^1(\mathbf{e}_h^i)| + |\Lambda_h^2(\mathbf{e}_h^i)|) \leq C(T, \epsilon) k^2 e^{2\alpha nk} + k\epsilon \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_h^i\|^2. \quad (5.52)$$

Observe that

$$|\Lambda_h^3(\mathbf{e}_h^i)| \leq |b(\mathbf{U}_H^i, \mathbf{e}_h^i, \mathbf{e}_h^i)| + |b(\mathbf{e}_H^i, \mathbf{u}_h^{*i}, \mathbf{e}_h^i)|. \quad (5.53)$$

With the help of Lemmas 3.1, 5.1, 5.3 and Young's inequality, it follows that

$$2k \sum_{i=1}^n e^{2\alpha ik} |\Lambda_h^3(\mathbf{e}_h^i)| \leq C(\epsilon) \sum_{i=1}^n e^{2\alpha ik} (\|\nabla \mathbf{e}_h^i\|^2 + \|\nabla \mathbf{e}_H^i\|^2) + \epsilon k \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_h^i\|^2. \quad (5.54)$$

For the estimation of the fourth term on the right hand side of (5.51), rewrite it as

$$|\Lambda_h^4(\mathbf{e}_h^i)| = |b(\mathbf{U}^i, \mathbf{e}_H^i, \mathbf{e}_h^i) - b(\mathbf{U}^i, \mathbf{e}_h^i, \mathbf{e}_h^i) + b(\mathbf{e}_h^i, \mathbf{u}_H^i - \mathbf{u}_h^{*i}, \mathbf{e}_h^i)|. \quad (5.55)$$

Apply Lemma 3.1, 4.1, 5.1, 5.5 and Young's inequality to obtain

$$2k \sum_{i=1}^n e^{2\alpha ik} |\Lambda_h^4(\mathbf{e}_h^i)| \leq C(T, \epsilon) k \sum_{i=1}^n e^{2\alpha ik} (\|\nabla \mathbf{e}_h^i\|^2 + \|\nabla \mathbf{e}_H^i\|^2) + \epsilon k \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_h^i\|^2. \quad (5.56)$$

A use of (5.52)-(5.56) in (5.50) leads to

$$e^{2\alpha nk} \|\mathbf{e}_h^n\|^2 + k \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_h^i\|^2 \leq Ck^2 (e^{2\alpha(n+1)k} + e^{2\alpha nk}) + Ck \sum_{i=1}^n e^{2\alpha ik} \|\mathbf{e}_h^i\|^2. \quad (5.57)$$

Use discrete Gronwall's Lemma to arrive at

$$\|\mathbf{e}_h^n\|^2 + ke^{-2\alpha nk} \sum_{i=1}^n e^{2\alpha ik} \|\nabla \mathbf{e}_h^i\|^2 \leq K_T k^2. \quad (5.58)$$

A use of (5.58) along with Theorem 3.1 completes the proof of error estimates for velocity in Theorem 5.1.

Using the similar techniques as to arrive at (5.49) and Theorem 3.1, the desired pressure estimate in Theorem 5.1 can be obtained and this will conclude the proof of Theorem 5.1. \square

6 Numerical Experiments

In this section, numerical results are presented to support theoretical results in Theorem 5.1. For space discretization, P_2 - P_0 mixed finite element space is used. We choose the domain $\Omega = (0, 1) \times (0, 1)$, time $t = [0, 1]$, coefficients $\nu = 1$ and $h = \mathcal{O}(H^2)$. Here, N denotes the number of unknowns in the system.

Example 6.1. *The right hand side function f is chosen in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is $u_1 = 2e^t x^2(x-1)^2 y(y-1)(2y-1)$, $u_2 = -2e^t y^2(y-1)^2 x(x-1)(2x-1)$, $p = ye^t$.*

Table 1 gives the numerical errors and convergence rates obtained on successively refined meshes for backward Euler scheme with $k = \mathcal{O}(h^2)$ applied to two grid system (3.3)-(3.5). The theoretical analysis provides a convergence rate of $\mathcal{O}(h^2)$ in \mathbf{L}^2 -norm, of $\mathcal{O}(h)$ in \mathbf{H}^1 -norm for velocity and of $\mathcal{O}(h)$ in L^2 -norm for pressure with a choice of $k = \mathcal{O}(h)$. These results support the optimal theoretical convergence rates obtained in Theorem 5.1

N	h	$\ \mathbf{u}(t_n) - \mathbf{U}^n\ $	Rate	$\ \mathbf{u}(t_n) - \mathbf{U}^n\ _{\mathbf{H}^1}$	Rate	$\ p(t_n) - P^n\ $	Rate
	1/4	0.009085		0.139927		0.548331	
577	1/8	0.002651	1.777183	0.075081	0.898156	0.281244	0.963220
2433	1/16	0.000713	1.893768	0.038833	0.951145	0.142265	0.983237
9986	1/32	0.000184	1.950443	0.019731	0.976861	0.071518	0.992191
40449	1/64	0.000046	1.976824	0.009940	0.989066	0.035856	0.996088

Table 3: Errors and convergence rates for backward Euler method with $k = \mathcal{O}(h^2)$.

Example 6.2. *In this example, we choose the right hand side function f in such a way that the exact solution $(\mathbf{u}, p) = ((u_1, u_2), p)$ is:*

$$u_1 = te^{-t^2} \sin^2(3\pi x) \sin(6\pi y), \quad u_2 = -te^{-t^2} \sin^2(3\pi y) \sin(6\pi x),$$

$$p = te^{-t} \sin(2\pi x) \sin(2\pi y).$$

In Table 2, we have shown the convergence rates for backward Euler method, respectively for \mathbf{L}^2 and \mathbf{H}^1 -norms in velocity and L^2 -norm in pressure with $k = \mathcal{O}(h^2)$. These results agree with the optimal theoretical convergence rates obtained in Theorem 5.1.

N	h	$\ \mathbf{u}(t_n) - \mathbf{U}^n\ $	Rate	$\ \mathbf{u}(t_n) - \mathbf{U}^n\ _{\mathbf{H}^1}$	Rate	$\ p(t_n) - P^n\ $	Rate
	1/4	0.132916		3.736491		0.989116	
577	1/8	0.028166	2.238442	1.537594	1.281009	0.186591	2.406260
2433	1/16	0.003717	2.921475	0.463199	1.730969	0.063322	1.559099
9986	1/32	0.000473	2.971736	0.124017	1.901084	0.016057	1.979463
40449	1/64	0.000063	2.894587	0.032022	1.953411	0.006437	1.318638

Table 4: Errors and convergence rates for backward Euler method with $k = \mathcal{O}(h^2)$.

References

- [1] Ait Ou Ammi, A. and Marion, M., *Nonlinear Galerkin methods and mixed finite elements: two-grid algorithms for the Navier-Stokes equations*, Numer. Math. **68** (1994), 189-213.
- [2] Garcia-Archilla, B. and Titi, E. S., *Postprocessing the Galerkin method: The finite element case*, SIAM J. Numer. Anal. **37** (2000), 470-499.
- [3] Abboud, H. and Sayah, T., *A full discretization of the time-dependent Navier-stokes equations by a two-grid scheme*, M2AN Math. Model. Numer. Anal. **42** (2008), 141-174.
- [4] Abboud, H., Girault, V. and Sayah, T., *A second order accuracy in time for a full discretized time-dependent Navier-Stokes equations by a two-grid scheme*, Numer. Math. **114** (2009) 189-231.
- [5] Dai, Xiaoxia and Cheng, Xiaoliang, *A two-grid method based on Newton iteration for the Navier-Stokes equations*, J. Comput. Appl. Math. **220** (2008), 566-573.
- [6] Fairag, F. A., *A Two-level Finite Element Discretization for the stream function form of the Navier-Stokes equations*, Comput Math Appl. **36** (1998), 117-127.
- [7] Frutos, J. de, Garc?a-Archilla, B. and Novo, J., *Optimal error bounds for two-grid schemes applied to the Navier-Stokes equations*, Appl. Math. Comput. **218** (2012), 7034-7051.
- [8] Girault, V. and Lions, J. L., *Two-grid finite-element schemes for the steady Navier-Stokes problem in polyhedra*, Portugal. Math. **58** (2001), 25-57.
- [9] Girault, V. and Lions, J. L., *Two-grid finite-element schemes for the transient Navier-Stokes problem*, M2AN Math. Model. Numer. Anal. **35** (2001), 945-980.
- [10] Heywood, J. G. and Rannacher, R., *Finite element approximation of the nonstationary Navier-Stokes problem: I. Regularity of solutions and second order error estimates for spatial discretization*, SIAM J. Numer. Anal. **19** (1982), 275-311.
- [11] Heywood, J. G. and Rannacher, R., *Finite element approximation of the nonstationary Navier-Stokes problem: IV: Error Analysis For Second-Order Time Discretization*, SIAM J. Numer. Anal. **27** (1990), 353-384.
- [12] Hill, A. T. and Süli E., *Approximation of the global attractor for the incompressible Navier-Stokes equations*, IMA J. Numer. Anal. **20** (2000), 633-667.
- [13] Xu, J., *Two-grid discretization techniques for linear and nonlinear PDEs*, SIAM J. Numer. Anal. **33** (1996), 1759-1777.

- [14] Xu, J., *A novel two-grid method for semilinear elliptic equations*, SIAM J. Sci. Comput. **15** (1994), 231–237.
- [15] Layton, W. and Tobiska, L., *A two level method with backtracking for the Navier-Stokes equations*, SIAM J. Numer. Anal. **35** (1998), 2035–2054.
- [16] Layton, W. and Lenferink, W., *Two-level Picard and modified Picard methods for the Navier-Stokes equations*, Appl. Math. Comput. **69** (1995), 263–274.
- [17] Layton, W. and Lenferink, H. W. J., *A multilevel mesh independence principle for the Navier-Stokes equations*, SIAM J. Numer. Anal. **33** (1996), 17–30.
- [18] Layton, W., *A two-level discretization method for the Navier-Stokes equations*, Comput. Math. Appl. **26** (1993), 33–38.
- [19] Niemistö, A., *FE-approximation of unconstrained optimal control like problems*, Report No. 70, University of Jyväskylä, (1995).
- [20] He, Y., *Two-level method based on finite element and Crank-Nicolson extrapolation for the time-dependent Navier-Stokes equations*, SIAM J. Numer. Anal. **41** (2003), 1263–1285.
- [21] Temam, R., *Navier-Stokes equations, theory and numerical analysis*, North-Holland, Amsterdam, 1984.